Strategic Experimentation On A Common Threshold

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Abstract

High effort is costly, but often it is worth maintaining an adequate level of effort if effort levels below an unknown threshold triggers a loss. Examples abound in environment preservation and quality maintenance. I use a dynamic experimentation game to explore the dynamic informational interactions among players who search for a common but unknown threshold. Players contribute to the rate of decline in the common effort level, and the game ends with a costly breakdown once the effort falls below the threshold. In the unique symmetric pure-strategy stationary Markov equilibrium, effort decreases gradually over time and settles asymptotically at a cutoff level, conditional on no breakdown. The cutoff depends on patience, the cost of the breakdown, and the prior distribution of the threshold but not on the number of players. The equilibrium outcome of the continuous-time model is approached by the outcome of a discrete time model when period length tends toward zero.

Keywords: Experimentation; Free-Riding; Markov Equilibrium; Information Externality; Spatial Learning

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1 Introduction

Learning through experimentation is a common practice in many aspects of the economy. There are situations in the economy where people need to learn about an unknown

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threshold, at which there is a drastic change in payoffs. Examples abound. Manufacturers attempt to satisfy some vague regulatory criteria when setting the quality of their products. Countries with similar environmental conditions face a threshold level of effort to control pollution, below which a costly environmental catastrophe occurs. The risk management in the banking sector struggles to search for the threshold level of risk exposure above which financial distress ensues, but low risk is costly in terms of profit.

In these examples, whenever the outcome, as a function of the current level of "effort," takes a discontinuous all-or-nothing form, the threshold that triggers the discrete change in outcome is an important parameter to learn. Moreover, while effort levels below the threshold are clearly undesirable (punishment from regulators, ecological disaster, financial distress, etc.), the ones far above the threshold are needlessly costly (higher quality, higher effort, less risky investment, etc.). This tension poses a non-trivial learning problem, raising the following questions: Do players always want to take the risk to reduce the level of effort? How do players interact in a collective experimentation situation? What is the long-run level of effort? How does the monitoring structure affect the equilibrium dynamics?

In the literature, a large number of models employ the bandit framework, whereby agents learn about the quality of one or several arms through a stochastic process (Keller, Rady, and Cripps (2005), Bolton and Harris (1999), Bonatti and Hörner (2011), etc.). However, the arms are typically assumed to be uncorrelated, so that there is no role for a "threshold arm" below which the arms are qualitatively different from all those above.

To answer the aforementioned questions, which the previous literature cannot address, I present a model that features a continuum of effort levels (arms). In my model, the arms are correlated in the following sense. If one effort level is below the threshold, then so are all lower effort levels. The continuum of arms poses a "spatial" dimension that is the main object of learning. In a typical exponential bandit problem (Keller, Rady, and Cripps (2005), for example), learning is slow in the sense that it takes time for belief to evolve regarding a specific arm. In some sense, this way of learning can be called "chronological" as the key decision is *when* to switch arms. In my model, in contrast, learning regarding a specific arm is assumed to be instant, but due to the continuous action space the search for the threshold remains slow. I call this "spatial" learning, because the key decision parameter is *how fast* to skim down the spectrum of arms.

Specifically, I first consider a multi-player cooperative problem as a benchmark for the first best. There is a common initial *effort level*. Time is continuous. At each instant, all players receive a fixed flow benefit, and choose a *contribution* to the decline of the common effort level. The higher the effort, the larger the flow cost. The process continues until the current common effort reaches the threshold. The threshold is fixed from the beginning and is *unknown* to the players. At that moment, the game ends with a breakdown, from which the team incurs a lumpy cost and receives a terminal payoff that is equal to the present discounted value of remaining at the threshold thereafter. As shown in Section 3, the solution to the team problem entails a very rapid decline in common effort as long as it is above a certain cutoff, after which it remains constant.

Having solved the cooperative problem for the planner, the main section investigates the strategic experimentation game. When the threshold is reached, the lumpy cost is shared among players proportionally to the individual contributions at that moment. There are information externalities among players: the history of play is common knowledge, and the thresholds they face are identical. Contribution from one player benefits the others by providing more learning about the threshold, whereas a larger contribution means a higher share in the lumpy cost, which is not internalized. In other words, a player learns from the others' contributions either when there is a breakdown (in which case the player avoids the lumpy cost herself), or when there is not (in which case the player can confidently operate at a lower effort level). As a result, players tend to contribute less to the decline in effort, thus free riding on the information provided by others. The game has a unique symmetric Markov equilibrium, and its dynamics differ qualitatively from those of the planner's solution: The common effort declines smoothly over time, faster in the beginning and then slowing towards a long-run asymptotic level. Interestingly, this asymptotic level is the same as the cutoff level in the team problem. The game also offers some comparative statics and implications that address the questions raised earlier.

The modeling choices of the continuous-time game is vindicated by the analysis of the discrete-time version of it. Specifically, I also consider a discrete-time game, indexed by period length, that features alternating moves, such that two players take turns to reduce the current effort level. As the period length tends toward zero, a pure-strategy symmetric Markov stationary equilibrium always exists, and the equilibrium outcome converges to that of the continuous-time game. This provides a robustness check: the formulation of the game in continuous time is indeed a convenient approximation of the discrete-time game.

The paper connects to several branches of existing literature. First, it belongs to the literature on learning and experimentation à la Rothschild (1974). Aghion, Bolton, Harris, and Jullien (1991), Agrawal (1995), Cope (2009) and Zhai, Tehrani, Li, Zhao, and Zhao (2011) studied statistical decision problems where the choice of arms is a continuum.

Games with strategic players facing exponential bandits (Keller, Rady, and Cripps (2005), Bonatti and Hörner (2011), Bonatti and Hörner (2015)) or Brownian bandits (Bolton and Harris (1999), Moscarini (2005)) have also been extensively studied, where the number of arms is finite. My work features the strategic aspect and a continuum of arms.

More broadly, the paper contributes to the study of dynamic games. Admati and Perry (1991), Marx and Matthews (2000), Compte and Jehiel (2003) and Matthews (2013) studied the dynamic problem of contribution to a public good. Recently, Iijima and Kasahara (2015) presented a framework for a class of dynamic contribution games in continuous time. In these papers, either the finishing line of the project is known, or there is no finishing line. Therefore, no learning takes place. In my model, however, the threshold of breakdown is unknown, and learning (with its externalities) plays an important role in determining the free-riding behavior.

Moreover, the paper is closely related to several papers on "spatial learning". Bonatti and Hörner (2013) considered a single-player learning problem with respect to an unknown threshold, where the action space is finite and the consequence of taking an excessively low action is a Poisson process. Callander (2011) featured spatial learning in which the realization of each trial is a sample point on a path of Brownian motion. Tsur and Zemel (1995, 1996) studied two decision problems with an unknown threshold in the context of resource extraction and greenhouse gas emission, respectively. Rob (1991) investigated a model in which a continuum of firms collectively learn the position of a kink in the demand curve through entry and exit. My work models a different underlying economic situation and investigates the strategic interaction between several far-sighted players learning the same object.

The remainder of the paper is organized as follows. Section 2 describes the set up. Section 3 considers the cooperative problem, serving as a benchmark. Section 4 solves the strategic problem for the multi-player game. Section 5 describes the discrete-time counterpart of the game and demonstrates the convergence of the solutions to that of the continuous-time game. Section 6 extends the game in three directions. Section 7 concludes.

2 Model Setup

The Game Time is continuous. The duration of the game is random, and the realized duration is denoted by $\bar{t}$. There are $I \geqslant 1$ players, labeled $i = 1, \ldots, I$. At each instant

$t \in \mathbb{R}_+$, Player i ($i = 1, \ldots, I$) chooses an action $v_i(t) \in [0, \overline{v}]$ (where $\overline{v} > 0$) if the game has not ended, and I write $v_i = \{v_i(t)\}_{t=0}^{\bar{t}}$ as the resulting path.

There is a state variable $x \geqslant 0$ that I interpret as the level of (collective) effort. The law of motion of x is as follows:

$$x(t) = x_0 - \int_0^t V(s)\mathrm{d}s, \quad \text{where } V(t) \equiv \sum_{i=1}^{I} v_i(t) \tag{1}$$

for some initial value $x_0 > 0$. Hence, each individual $v_i(t)$ serves as an additive contribution to the total "rate of decline" in x.

The game features incomplete information. Specifically, there is a random state of the world $c \in [0, x_0)$. It is distributed with c.d.f. $F(\cdot)$ and a continuous density $f(\cdot)$ on the support $[0, x_0]$. The realized value of c is fixed throughout, but unknown at the start. The game ends when $x(t) \leqslant c$ for the first time, if ever. In this sense, c is regarded as a *threshold*.[1] The duration of the game is hence a stopping time:

$$\bar{t} = \inf\{t : x(t) \leqslant c\} \in \mathbb{R}_+ \cup \{+\infty\}.$$

Because $V(t)$ is bounded for any $t \in [0, \bar{t}]$, the time path of effort $x(\cdot)$ is continuous. Therefore, if $\bar{t} < \infty$, the threshold c is revealed to be $x(\bar{t})$ at the end of the game.

Henceforth, I maintain the following assumptions concerning the distribution of c, unless explicitly mentioned otherwise.

Assumption 1 *(Monotone Hazard Rate)*
The inverse hazard rate $\frac{F(\cdot)}{f(\cdot)}$ is strictly increasing.

Assumption 2 *(Strongly Positive Density)*
The density $f(\cdot)$ is uniformly bounded away from zero, i.e. $\exists \underline{f} > 0$ s.t. $f(c) \geqslant \underline{f}$ for all $c \in [0, x_0]$.

Assumption 3 *(Lipschitz Continuous Density)*
The density $f(\cdot)$ is Lipschitz continuous, i.e., $\exists \kappa > 0$ s.t. $|f(x) - f(y)| \leqslant \kappa|x - y|$ for all $x, y \in [0, x_0]$.

Assumption 1 is a standard monotonicity assumption. Assumptions 2 and 3 rule out pathological solutions of the game.

[1] In the language of bandits, c is the threshold that divides the continuum of arms into "good" arms ($x > c$) and "bad" arms ($x \leqslant c$).

Payoff comes in two forms: flow and lump. The flow payoffs consist of both cost and benefit. A higher flow cost results from higher effort. With little loss of generality, the flow cost is simply x for all players when the effort level is x.[2] Moreover, each player enjoys a fixed flow benefit $p > x_0$ at all times.

Lumpy payoffs come at the end of the game, if it ends in finite time. Given a fixed path $x(\cdot)$, the game ends at $\bar{t}$ ($\bar{t} = \infty$ means no breakdown) along with a costly *breakdown*. Let r be the common discount rate of the players. First, the breakdown brings lumpy cost $L > 0$, if ever, at time $\bar{t}$. It is assumed that $Lr < \frac{F(x_0)}{f(x_0)}$, i.e., the lumpy cost is relatively small. [3] The lumpy cost is divided among players as follows: Player i suffers the loss $L\frac{v_i(\bar{t})}{V(\bar{t})}$, such that the share of cost equals the share of actions at that moment.[4,5] Second, each player receives a terminal benefit $\frac{p-c}{r} > 0$.[6]

In sum, if the breakdown occurs at time $t = \bar{t}$, which reveals threshold $c = x(\bar{t})$, then Player i's realized total present discounted payoff is

$$\underbrace{\int_0^{\bar{t}} (p - x(t))\, e^{-rt} \mathrm{d}t}_{\text{flow before } \bar{t}} + \underbrace{\frac{p-c}{r} e^{-r\bar{t}}}_{\text{terminal benefit at } \bar{t}} - \underbrace{L \frac{v_i(\bar{t})}{V(\bar{t})} e^{-r\bar{t}}}_{\text{terminal cost at } \bar{t}},$$

where $x(t)$ is determined by (1).

Information and Beliefs I assume perfect monitoring of players' actions. A *plausible* history of length t is denoted by $h^t \equiv \{v_i(s) : s < t\}_{i=1}^I$ such that $x(t) > 0$, and the set of

[2] As long as the flow cost function $\phi(x)$ is Lipschitz continuous and strictly increasing in x, we can always redefine the state variable (the effort level) to be $\phi(x)$ instead of x, such that the cost is again equal to the state.

[3] As will become evident later, under this assumption, at least some learning is worthwhile at the beginning.

[4] It is well defined because the breakdown arrives only when $V > 0$.

[5] Some explanation is in order concerning why the probability of suffering the lumpy cost is shared in this way. It is not some designed "cost-sharing rule" that players agree upon; it is simply a natural implication if we view the continuous-time game as a limit of discrete-time games with very short period length. In the latter, players take turns to bring x downward from the previous level. If the turns shift frequently enough, then the probability of reaching the threshold during one's turn is approximately proportional to the change in x during one's turn. The reader is referred to Section 5 for a description of a discrete-time model.

[6] This form of the terminal payoff is not crucial for the results. It is feasible to allow fairly general functions, but the one used here has the following economic interpretation: the arrival of breakdown at $\bar{t}$ fully reveals the location of the threshold, and hence if there were a continuation game after the stopping time, the supremum of the payoff for any player is a constant flow of $p - c$. The terminal benefit $\frac{p-c}{r} > 0$ is the present discounted value of this flow. It is positive because by assumption $p > x_0 \geqslant c$. One can also view this as a normalization because in this way the real cost of a breakdown is indeed L, instead of L plus some change in the continuation value.

all plausible histories of length t is $\mathcal{H}^t$. The space of all plausible histories is $\mathcal{H} \equiv \cup_{t\in\mathbb{R}_+}\mathcal{H}^t$. Beginning with the common prior $F(\cdot)$ at $t = 0$, players update their beliefs regarding the distribution of the threshold, provided that no breakdown has occurred thus far. Because for every x the outcome $\mathbb{1}_{\{x>c\}}$ is revealed immediately, the updating process is a mere truncation at the top of the prior distribution.

Solution Concept A pure strategy of Player i is a measurable mapping from $\mathcal{H}$ into $[0, \overline{v}]$. I focus on stationary Markov perfect equilibrium in pure strategies, and hence the strategy profile depends only on the payoff-relevant state variable $x \in (0, x_0]$ but not on calendar time or on how the state x is reached. Formally, for all $i = 1, \ldots, I$, a stationary Markov strategy for Player i is a measurable mapping $\nu_i : (0, x_0] \to [0, \overline{v}]$. A profile of Markov strategies is denoted by $\nu \equiv \{\nu_i\}_{i=1}^I$. The time path of x follows the ordinary differential equation below:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sum_{i=1}^{I} \nu_i(x), \quad x(0) = x_0. \tag{2}$$

To ensure the existence and uniqueness of $x(\cdot)$, I restrict attention to strategies such that for all $i = 1, \ldots, I$: (a) ν_i is left continuous for all $i = 1, \ldots, I$, and (b) ν_i is piecewise Lipschitz continuous with at most finitely many jumps. With these restrictions on the strategy profile, the path of x is uniquely determined by ν, and thus, the stopping time $\bar{t}$ is well defined.

Given the strategy profile of other players, Player i aims to maximize the expected payoff

$$w_i(\nu) = \mathbb{E}\left[\int_0^{\bar{t}} (p - x(t))\, e^{-rt}\mathrm{d}t + \frac{p-c}{r}e^{-r\bar{t}} - L\frac{\nu_i(c)}{\mathcal{V}(c)}e^{-r\bar{t}}\right], \tag{3}$$

where $x(t)$ is determined by ν via (2), $\bar{t}$ is pinned down by both ν and c, and the expectation is taken over c. A stationary Markov equilibrium in pure strategies is a profile ν that constitutes a Nash equilibrium for any initial state $x \in (0, x_0]$.

3 Cooperative Problem

In this section, I discuss the cooperative problem as a benchmark for the first best. The Hamilton-Jacobi-Bellman equation (henceforth, HJB) of the social planner is given. Its analysis yields the solution to the problem, summarized in Proposition 1. Several implications from the result are discussed afterwards.

Denote $\overline{V} \equiv I\overline{v}$ as the upper limit on aggregate action, and denote $U_I(x)$ as the per-capita value function in the I-player cooperative problem. Formally, I solve the following optimization problem.

$$\begin{aligned} U_I(x_0) &= \frac{1}{I}\max_{\nu(\cdot)}\sum_{i=1}^{I} w_i(\nu) \\ &= \max_{\nu(\cdot)}\mathbb{E}\left[\int_0^{\bar{t}}(p-x(t))\,e^{-rt}\mathrm{d}t + \frac{p-c}{r}e^{-r\bar{t}} - \frac{L}{I}e^{-r\bar{t}}\right], \end{aligned}$$

such that x evolves according to (2), $\bar{t}$ is equal to the stopping time determined by x, and $\nu_i(x) \in [0, \overline{v}]$ for all $x \in [0, x_0]$. Because both x and $\bar{t}$ solely depend on $\mathcal{V}(\cdot) = \sum_{i=1}^{I}\nu_i(\cdot)$, it is sufficient for the social planner to control the aggregate speed $\mathcal{V}(\cdot)$ only.

Recall that with the assumption on L, we have $\frac{F(x_0)}{f(x_0)} > Lr \geqslant \frac{Lr}{I}$. On the other hand, $\lim_{x\to 0}\frac{F(x)}{f(x)} = 0 < \frac{Lr}{I}$.[7] By the Intermediate Value Theorem and Assumption 1, there exists a unique solution to the equation $\frac{F(x)}{f(x)} = \frac{Lr}{I}$. Let x_I^* denote the solution, where the subscript indicates the number of players in the problem.

For $x \in (0, x_0]$, if U_I is differentiable at x (as will be verified), then the HJB equation for the cooperative problem is

$$rU_I(x) = (p-x) + \max_{V\in[0,\overline{V}]} V\left\{\frac{f(x)}{F(x)}\left(\frac{p-x}{r} - U_I(x) - \frac{L}{I}\right) - U_I'(x)\right\}. \tag{4}$$

The first term on the right-hand side is the flow payoff (benefit and cost) per capita. The second term is the maximum of a linear function of V, and the bracket multiplying V summarizes the benefit and cost of taking aggregate action V. The first term in the bracket is the expected loss upon reaching the threshold, equalling the hazard rate $f(x)/F(x)$

[7] The equality holds for the following reason. By Assumption 1 $\frac{F(x)}{f(x)}$ is increasing, and by definition it is bounded below by 0. Hence, $\lim_{x\to 0}\frac{F(x)}{f(x)}$ exists and is non-negative. If $\lim_{x\to 0}\frac{F(x)}{f(x)} = a > 0$, then $f(x) \leqslant F(x)/a$ for all $x \geqslant 0$, and with initial condition $F(0) = 0$, we obtain $F(x) = 0$ for all $x \geqslant 0$, a contradiction. Hence, $\lim_{x\to 0}\frac{F(x)}{f(x)} = 0$.

times the net loss per capita conditional on breakdown. The second term in the bracket captures the gains from learning by reducing x to $x - V\mathrm{d}t$, when there is no breakdown in the next moment.

The linearity of the HJB leads to a corner solution: $V = 0$ or $V = \overline{V}$. When $V = 0$, we have $U_I(x) = (p - x)/r$. When $V = \overline{V}$, (4) implies the following ordinary differential equation:

$$\overline{V}U_I'(x) + \left(r + \frac{\overline{V}f(x)}{F(x)}\right)U_I(x) = p - x + \frac{\overline{V}f(x)}{F(x)}\left(\frac{p-x}{r} - \frac{L}{I}\right),$$

the solution to which is

$$U_I(x) = \frac{p-x}{r} + \frac{e^{-rx/\overline{V}}}{rF(x)}\left[C_1 + \int_0^x \left(F(s) - \frac{Lr}{I}f(s)\right)e^{rs/\overline{V}}\mathrm{d}s\right], \tag{5}$$

for some $C_1 \in \mathbb{R}$. The right-hand side of the above consists of two parts. The first term is the payoff of staying at x forever (no learning), and the second term is the option value of learning. Value-matching together with smooth-pasting pins down the constant C_1 and the cutoff effort level at which V switches from $\overline{V}$ to zero.

Proposition 1 *In the cooperative problem, the policy and the per-capita value function are*

$$V = \begin{cases} \overline{V} & \text{if } x \in (x_I^*, x_0] \\ 0 & \text{if } x \in (0, x_I^*] \end{cases},$$

$$U_I(x) = \begin{cases} \frac{p-x}{r} + \frac{1}{rF(x)}\int_{x_I^*}^x \left(F(s) - \frac{Lr}{I}f(s)\right)e^{-r(x-s)/\overline{V}}\mathrm{d}s & \text{if } x \in (x_I^*, x_0] \\ \frac{p-x}{r} & \text{if } x \in (0, x_I^*] \end{cases}. \tag{6}$$

Proof. The value-matching condition requires $U_I(x_I^*) = (p - x_I^*)/r$, and the smooth-pasting reduces to $U_I'(x_I^*) = -1/r$. Plugging these two equations into (5) yields $\frac{F(x_I^*)}{f(x_I^*)} = \frac{Lr}{I}$. The optimality of the solution is guaranteed by the Verification Theorem. ■

The solution (6) has several implications. First, there exists an inaction region (or "safety buffer") $[0, x_I^*]$. The fact that $x_I^* > 0$ means that the planner wants players to stop learning before they search through the entire spectrum of effort levels, even in the absence of a breakdown. Intuitively, there are both cost and benefit associated with the aggregate speed V. In (4), the benefit $-VU_I'(x)$ represents the possible future gains from learning conditional on the event that the threshold is not reached in the next moment, and the cost $\frac{Vf(x)}{F(x)}\left(\frac{p-x}{r} - U_I(x) - \frac{L}{I}\right)$ reflects the risk of incurring the lumpy cost from breakdown in the next moment. As x decreases, the hazard rate $f(x)/F(x)$ increases, and eventually, the cost outweighs the benefit. Intuitively, knowing that the range of the costly threshold is sufficiently narrowed down, the planner optimally calls off learning.

Second, the first best allows for no procrastination in learning. In other words, the planner wants the players to reach x_I^* as fast as possible and then stop learning immediately once it is not worthwhile. Procrastination serves to delay the arrival of breakdown but also necessitates a longer duration of high flow cost from effort. In the absence of breakdown, the time path of effort x is piecewise linear: skimming down from x_0 to x_I^* with maximum speed, followed by a constant level at x_I^* forever. As we will see in Section 4, procrastination naturally arises in non-cooperative situations.

Third, the reason that players switch from "risky" actions $V > 0$ to the "safe" action $V = 0$ differs from the one that applies to exponential bandit models. Consider an exponential bandit problem with a safe arm and a risky arm. If the risky arm features conclusive good news, then players eventually switch from the risky arm to the safe arm once they become sufficiently pessimistic, in the sense that the current sacrifice in the flow cost grows excessive relative to the potential benefit from learning. If, instead, the risky arm potentially generates conclusive bad news, then there is no voluntary switch from the risky arm to the safe one unless learning is stopped by a breakdown (Keller and Rady (2015)). The spatial learning model considered here differs from both cases. It is by nature a bad news model, and as the effort level decreases without triggering the breakdown, players grow more optimistic regarding the distribution of c in the sense of first-order stochastic dominance. However, unlike the exponential bandit with bad news, the learning stops voluntarily when players become sufficiently optimistic. The seemingly paradoxical solution is explained by the fact that while the posterior distribution truncated at a lower state is "better" for the planner, the hazard rate of a breakdown is larger for the lower state thanks to Assumption 1. Hence, it is the ordering of hazard rate that governs the incentives, rather than the first-order stochastic comparison of distributions. In this way, the model shares some properties of a good-news exponential bandit model, as learning decreases the state which in turn translates into lower incentives to keep learning, regardless of whether the value function is increasing or decreasing in the state. At some point, players stop learning as it becomes relatively too risky to search further down, even though the current effort level is considered to be cheap enough.

One can also interpret the stopping property from the perspective of a bandit problem with a continuum of correlated arms. Suppose that a social planner faces a current state variable x, which is the highest arm with uncertain output. Heuristically, consider operating a "bunch of arms" in the interval $[x + \mathrm{d}x, x]$ for the next instant of time $\mathrm{d}t$. This bunch

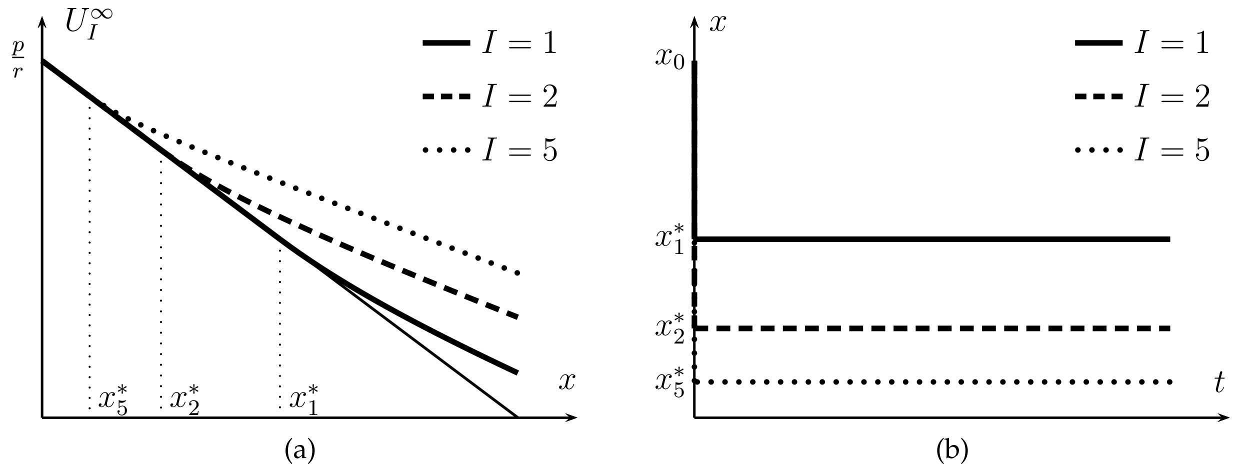

Figure 1: Cooperative Solution. (a) Value functions for 1, 2 or 5 players. (b) Time paths for 1, 2 or 5 players.

of arms, as a whole, is bad with probability $-\mathrm{d}x\frac{f(x)}{F(x)}$. Hence, the experimentation generates a good outcome (no breakdown) with probability $1 + \mathrm{d}x\frac{f(x)}{F(x)}$ which saves a future cost of $I\frac{-\mathrm{d}x}{r}$ in total, and generates a bad outcome (breakdown) with probability $-\mathrm{d}x\frac{f(x)}{F(x)}$ which inflicts a damage of L. This marginal experimentation is profitable if and only if $\frac{F(x)}{f(x)} > \frac{Lr}{I}$. In summary, the cost-benefit analysis for this bunch of arms is similar to that of an exponential bandit with bad news, but the situation in this model features a spectrum of such bunches (bad arms) with different priors of being bad, and there is a threshold bunch operating below which is not profitable.

Fourth, the cutoff effort level x_I^* for inaction is decreasing in the number of players. When $I = 1$, the special case of a single player obtains, and the cutoff effort x_1^* is higher than that for multiple players. This makes sense because the lumpy cost is shared among I players. From the perspective of a member in the team, having more members is beneficial in two aspects. One is intensive in that the lumpy cost is shared by more players. The other is extensive in that more learning (lower x_I^*) takes place in larger groups because the information from learning is shared to a greater extent.

Finally, the myopic cutoff effort is x_0. Myopic players care about only the payoff in the next moment. In this model, this payoff is proportional to $p - x + \frac{Vf(x)}{F(x)}\left(\frac{p-x}{r} - U_I(x) - \frac{L}{I}\right)$, strictly decreasing in V because $U_I(x) \geqslant \frac{p-x}{r}$. Hence, myopic players should never search downward because it only serves to harm this payoff.

It is interesting to regard the limit of the solution as $\overline{V} \to \infty$. While $\overline{V}$ approaches infinity, the time needed for the team to reach x_I^* shrinks to zero. In the limit, no time is taken to achieve the preferred amount of learning. Moreover, for any fixed x, the value function monotonically converges to

$$\begin{aligned} U_I^\infty(x) &\equiv \lim_{\overline{V}\to\infty} U_I(x) \\ &= \begin{cases} \frac{p-x}{r} + \frac{1}{rF(x)} \int_{x_I^*}^{x} \left(F(s) - \frac{Lr}{I} f(s)\right) \mathrm{d}s & \text{if } x \in (x_I^*, x_0] \\ \frac{p-x}{r} & \text{if } x \in (0, x_I^*] \end{cases}, \end{aligned}$$

which is the supremum of value functions for all $\overline{V}$. This limit function is interpreted as the per capita value achievable if learning requires no time.

Figure 1 summarizes the solution for the cooperative problem. The left panel shows the per capita value functions $U_I^\infty(x)$ for $I = 1$ (thick solid curve), $I = 2$ (dashed curve) and $I = 5$ (dot-dash curve). The thin solid line is the payoff from not experimenting at all. The three curves touch the thin solid line at x_1^*, x_2^* and x_5^*, respectively. The right panel shows the time path of x starting from x_0 when $\overline{V} \to \infty$, with 1, 2 or 5 players. After a rapid initial decline, they stop at the corresponding cutoff effort levels.

4 Strategic Problem

In a game among $I \geqslant 2$ players, incentives depend on positive information externalities, resulting in very different equilibrium dynamics. In approaching the problem, the HJB of a player is provided, and the best response correspondence is derived. Properties shared by all stationary Markov equilibria are then derived. As the focus is on symmetric equilibrium, the main result (Theorem 1) solves for the unique symmetric stationary Markov equilibrium in pure strategies in closed form, in which learning is slow relative to the social optimum. Some remarks and testable implications follow.

4.1 Best Response Function

For any strategy profile $\{v_i\}_{i=1}^I = \{\nu_i(x)\}_{i=1}^I$, define $W_i(x)$ as the value function for Player i, taking as given the strategy profile of all other players. Let $\nu_{-i}(x) \equiv \sum_{j\neq i} \nu_j(x)$ be the aggregate action contributed by all other players and $\mathcal{V}(x) \equiv \sum_{i=1}^I \nu_i(x)$ be the aggregate action of all players. The HJB for Player i is

$$\begin{aligned} rW_i(x) &= (p-x) + \max_{v_i \in [0,\overline{v}]} v_i \left\{ \frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) - L \right) - W_i'(x) \right\} \\ &\quad + \nu_{-i}(x) \left\{ \frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) \right) - W_i'(x) \right\}. \end{aligned} \tag{7}$$

The first term on the right-hand side is still the flow payoff. The second term (that with the maximum operator) is decomposed as the loss from breakdown and the benefit from learning generated by Player i's *own* action. The term in the second line arises only in the non-cooperative problem; it consists of the loss and benefit generated by *other* players. This HJB equation differs from that of the cooperative problem (or single-player problem) in an important way, in that the information provided by other players' actions benefits Player i with more learning and a lower probability of triggering the breakdown. This type of information externality can be described by "learning from others' (lack of) mistakes."

A strategic player who is concerned about the second term in (7) only faces a problem that is linear in v_i, and hence the shape of W_i completely determines the best response. The aggregate action ν_{-i} from other players affects W_i and thus serves indirectly as the argument for the best response:

$$BR_i(\nu_{-i}) \begin{cases} = \overline{v} & \text{if } \frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) - L \right) > W_i'(x), \\ \in [0,\overline{v}] & \text{if } \frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) - L \right) = W_i'(x), \\ = 0 & \text{if } \frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) - L \right) < W_i'(x). \end{cases} \tag{8}$$

A pure strategy stationary Markov equilibrium requires that $\nu_i(x) \in BR_i(\nu_{-i})$ for all $x \in (0, x_0]$. Proposition 2 states that in any pure strategy stationary Markov equilibrium of an I-player game, the amount of learning must lie between those of a single-player problem and an I-player cooperative problem.

Proposition 2 *In any pure strategy stationary Markov equilibrium, (a) $\mathcal{V}(x) > 0$ for $x > x_1^*$ and (b) $\mathcal{V}(x) = 0$ for $x < x_I^*$.*

Proof. See Appendix. ■

4.2 Symmetric Equilibrium

A pure strategy stationary Markov equilibrium is symmetric if $\nu_i(\cdot) = \nu(\cdot)$ for all $i = 1, \ldots, I$. As a result, symmetry implies that the value functions satisfy $W_i(\cdot) = W(\cdot)$. The following Proposition solves for the cutoff effort level where the actions switch from positive to zero.

Proposition 3 *In any symmetric pure strategy MPE, $\nu(x) > 0$ if $x > x_1^*$, and $\nu(x) = 0$ if $x < x_1^*$.*

Proof. See Appendix. ■

Proposition 3 indicates that an I-player game leads to the same inaction region $[0, x_1^*]$ as does a single player problem, which is clearly larger than the planner's optimal inaction region $[0, x_I^*]$. In other words, if a single player optimally decides to maintain the current effort level, then adding another player will not prompt her to reduce her effort level if we restrict attention to symmetric equilibrium. Here is the intuition. Suppose with multiple players, the cutoff effort level is lowered to $\hat{x} < x_1^*$. Is it sequentially rational for Player i to maintain this stopping rule when facing a state slightly above $\hat{x}$? No. In such a state, learning will stop soon anyway, and the cost of experimentation is approximately $\frac{f(\hat{x})}{F(\hat{x})} v_i L > \frac{f(x_1^*)}{F(x_1^*)} v_i L = \frac{v_i}{r}$, while the benefit from experimentation is approximately $-v_i W_i'(\hat{x}) = \frac{v_i}{r}$. Hence, the cost exceeds the benefit, contradicting optimality for Player i.

The cutoff action only informs us that the *eventual amount* of the decrease in effort is insufficient in equilibrium, relative to the first best. It is also of interest to consider the *speed* of the decrease in effort because slow decrease means remaining at high effort levels for a longer period. The following theorem, the main result of the section, characterizes the unique symmetric equilibrium of the game, where the decline in effort is indeed slow.

Theorem 1 *Suppose that $\overline{v} \geqslant \frac{\int_{x_1^*}^{x_0} (F(s) - Lrf(s)) \mathrm{d}s}{(I-1)L\underline{f}}$. There exists a unique symmetric pure strategy stationary Markov equilibrium in the I-player strategic problem. Furthermore, the equilibrium features*

$$\nu(x) = \begin{cases} \frac{\int_{x_1^*}^{x} (F(s) - Lrf(s)) \mathrm{d}s}{(I-1)Lf(x)} & \text{if } x \in (x_1^*, x_0] \\ 0 & \text{if } x \in (0, x_1^*] \end{cases}, \tag{9}$$

$$W(x) = U_1^\infty(x). \tag{10}$$

Proof. See Appendix. ■

The condition $\overline{v} \geqslant \frac{\int_{x_1^*}^{x_0}(F(s)-Lrf(s))\mathrm{d}s}{(I-1)L\underline{f}}$ in the theorem is not necessary for existence and uniqueness; it is imposed to avoid corner solutions for $\nu(\cdot)$. There are two noteworthy remarks. First, in the unique symmetric equilibrium, the individual action $v = \nu(x)$ is an interior solution, in contrast with the planner's problem. This means that the players are indifferent among any action in the range $[0, \overline{v}]$ for $x \in (x_1^*, x_0]$ but choose the specific action as part of the symmetric equilibrium.

Second, the indifference among actions implies rent dissipation. Given the equilibrium strategies of other players, one can do equally well by choosing the highest action $\overline{v}$. As $\overline{v} \to \infty$, the difference between the payoff of this player in the game and payoff when playing alone vanishes (equation (10)), demonstrating an extreme form of rent dissipation. Although having other players in the game seemingly benefits a player, the procrastination in learning and the resulting high flow cost almost entirely negates the rent.

Having discussed the equilibrium strategies, it is interesting to elaborate on the behavior of the time path of effort. Absent any breakdown, the path $x(\cdot)$ is uniquely determined by the differential equation (2). From Theorem 1, we know that x decreases over time as long as $x > x_1^*$ and never reaches a state below x_1^*. Moreover, as is evident from (9), the individual action (and also the aggregate action) shrinks to zero when $x \downarrow x_1^*$, reflecting a severe shading in actions when the effort level is close to the cutoff. Proposition 4 below demonstrates that the state never reaches x_1^*.

Proposition 4 *When $c \leqslant x_1^*$, the time path $x(t)$ satisfies (a) $\lim_{t\to\infty} x(t) = x_1^*$ and (b) $x(t) > x_1^*$ for all $t \in \mathbb{R}_+$.*

Proof. First, note that by Assumption 3, f defined on $[0, x_1^*]$ must have an upper bound $\overline{f} > 0$.

Part (a): $x(t)$ is non-increasing and is bounded below by x_1^*. If $\lim_{t\to\infty} x(t) = \hat{x} > x_1^*$, then by Assumption 1, $x'(t) = -I\nu(x) < -\frac{I\int_{x_1^*}^{\hat{x}}[F(s)-Lrf(s)]\mathrm{d}s}{(I-1)L\overline{f}} < 0$ for all $t \in \mathbb{R}_+$, a contradiction.

Part (b): $\nu(x)$ is bounded above by $\hat{\nu}(x) \equiv \frac{\int_{x_1^*}^{x}(\overline{f}+Lr\kappa)(s-x_1^*)\mathrm{d}s}{(I-1)L\underline{f}}$. The path of x, when ν is replaced by $\hat{\nu}$, is always above x_1^*, and hence this must also be true for the original path.
■

The proposition states that absent breakdowns, the long-run limit of x is x_1^*, and hence the probability of eventually learning the threshold is $1 - F(x_1^*)$. It also implies that for

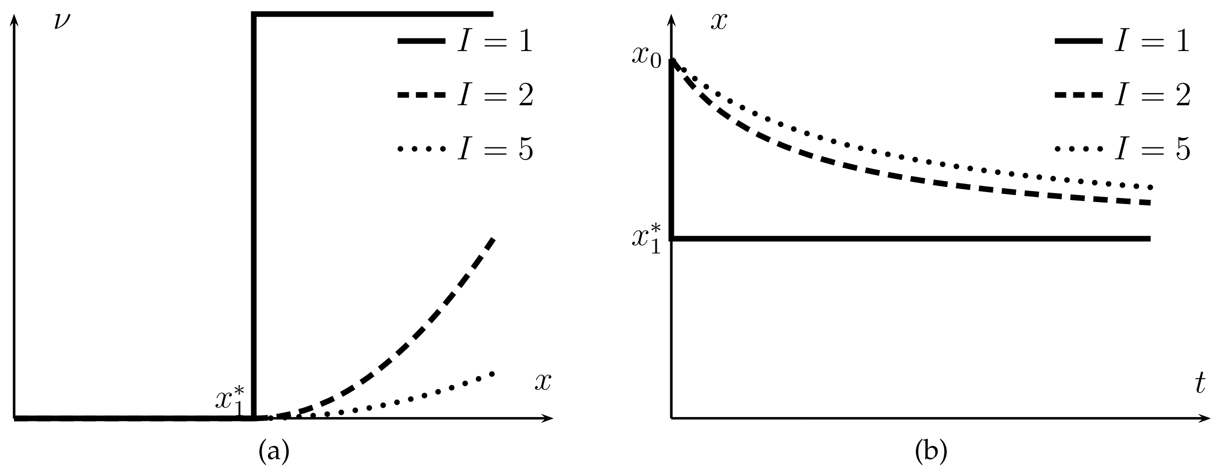

Figure 2: Symmetric Equilibrium. (a) Equilibrium strategy for 1, 2 or 5 players. (b) Time paths for 1, 2 or 5 players.

threshold realizations slightly above x_1^*, the time to learn its exact location becomes unbounded. It is possible to derive the speed of convergence as follows.

Proposition 5 *If* $\lim_{x \downarrow x_1^*} \frac{F(x)/f(x) - Lr}{x - x_1^*} = b > 0$, *then conditional on no breakdown, the time path* $x(t)$ *converges to* x_1^* *at speed* t^{-1}. *Specifically,*

$$\lim_{t\to\infty} \frac{x(t) - x_1^*}{1/t} = \frac{2(I-1)L}{bI}.$$

Proof. See Appendix. ■

Figure 2 shows the equilibrium strategy $v_i = \nu_i(x)$ and the time path for games with $I = 2, 5$ players. As a comparison, the time path for the single-player problem is also presented.

4.3 Comparative Statics and Implications

The dynamics of the game depend on the primitives. Fix $x_0 > 0$; I first examine the effect on the inaction region of changing the discount rate, the lumpy cost and the prior distribution function.

Corollary 1 *The cutoff effort level* x_1^* *is increasing in* L *and* r. *Moreover, if two prior distribution functions* F *and* F' *are hazard-rate ordered such that* $\frac{f(x)}{F(x)} < \frac{f'(x)}{F'(x)}$ *for all* $x \in [0, x_0]$, *then* x_1^* *under* F *is lower than that under* F'.

Proof. By Assumption 1, the above comparative statics are immediate from the equation $\frac{F(x_1^*)}{f(x_1^*)} = Lr$. ■

The cutoff is directly related to the probability of eventual learning $1 - F(x_1^*)$. Hence, projects with a less-costly breakdown operated by a more-patient group will likely have a lower cutoff effort level and a higher probability of eventual learning. As players become perfectly patient, sufficient learning is achieved. The second part of Corollary 1 states that if the prior distribution is more favorable (concentrated more in the lower end), then the cutoff is lower.

The model also generates some comparative statics regarding the equilibrium actions. However, these results have testable implications only if the action history is also available to an outside econometrician.

There are contexts in which the action/effort history is observable. Examples include leverage levels in financial sectors and the amount of pollution in the environment. In these situations, we have the following predictions.

Lemma 1 *Fix distribution F. At any state $x > x_1^*$, the individual action $\nu(x)$ (as well as the aggregate action $\mathcal{V}(x)$) is decreasing in L, r, and I.*

Proof. See Appendix. ■

The lemma implies that (a) in scenarios with more severe consequences of a breakdown, players are less eager to reduce the effort level; (b) the same is true if the group of players is less patient; and (c) the aggregate action (the rate of the decline in effort) is decreasing in the number of players I.

While statement (a) is somewhat expected, (b) and (c) deserve brief comments. Statement (c) implies that the negative impact of free-riding outweighs the number of players, and accordingly, even the aggregate action is decreasing in I. As $I \to \infty$, the aggregate action becomes half of that in a two-player game. Statement (b) is surprising at first glance if one fits the story into a "quality maintenance" context. Why should less-patient players keep a higher quality (effort) level? The reasoning is as follows. The lumpy cost occurs immediately when $x < c$, and hence, discounting plays no role in the effectiveness of the "punishment" that follows. Therefore, the risk of lumpy cost serves as part the current payoff whereas the effort level, on the other hand, contributes to the continuation payoff because of the option value of learning[8]. It is precisely the impatient players who do not

[8]In a typical repeated game model without learning, the role of "effort level" and "punishment" is swapped. The effort level is committed for a period and serves as part of the current payoff. The punish-

wish to learn more at the expense of the current risk.

There are more situations in which the action/effort history is unobservable or difficult to quantify. For instance, product qualities from manufacturers are unobservable, and maintenance effort levels for power plants are difficult to measure. Thus, one cannot use the action or effort path to make predictions. However, there is an implication based on the hazard rate: the hazard rate of breakdown is decreasing over time. It is useful for an outside econometrician as long as the occurrence of incidence (breakdown) is observable.

Lemma 2 *The hazard rate of breakdown $I\nu(x)\frac{f(x)}{F(x)}$ is decreasing over time.*

Proof. To see this, note that the hazard rate of breakdown is

$$I\nu(x)\frac{f(x)}{F(x)} = \frac{I\int_{x_1^*}^{x}(F(s) - Lrf(s))\mathrm{d}s}{(I-1)LF(x)},$$

and this is increasing in x because

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}\left(I\nu(x)\frac{f(x)}{F(x)}\right) > 0 \\ \Leftrightarrow \quad & \left(\frac{F(x)}{f(x)} - Lr\right)F(x) > \int_{x_1^*}^{x}\left(\frac{F(s)}{f(s)} - Lr\right)f(s)\mathrm{d}s, \end{aligned}$$

which is true by Assumption 1. ■

Hence, as time passes, x decreases, and the hazard rate of triggering a breakdown is decreasing. This coincides with the general observation that long-established firms tend to maintain a steady quality level and are less likely to suffer from scandals involving quality issues.

5 Discrete Time and Convergence

This section studies a discrete-time version of the game. In particular, the game features alternating moves where $I \geqslant 2$ players take turns and potentially change the current effort to a lower level. A breakdown arrives as soon as the effort level falls below the unknown threshold after some player's move, and that player alone bears the lumpy cost of

ment comes at least one period later and serves as part of the continuation payoff.

the breakdown. The breakdown also ends the game and yields a terminal payoff, just as in the continuous-time version.

The purpose of this section is to let the frequency of moves go to infinity (i.e., period length goes to zero) and to demonstrate that the realized time path of effort in the discrete-time game converges to that of the continuous-time version. In this sense, the outcome of the continuous-time game is robust to perturbations in the fineness of time grids.

Formally, time is discrete, and the time periods are indexed by $n = 1, 2, \ldots$. The horizon of the game is random. The period length is $\Delta > 0$. For simplicity, let there be $I = 2$ players labeled $i = 1, 2$. In each of the odd periods n, Player 1 chooses an action $v_1(n) \in [0, \overline{v}]$ (again, $\overline{v} > 0$), provided that the game has not ended. In even periods, Player 2 chooses an action $v_2(n) \in [0, \overline{v}]$. The effort level x evolves as follows:

$$x(n) = \begin{cases} x(n-1) - v_1(n)\Delta & \text{for odd } n, \\ x(n-1) - v_2(n)\Delta & \text{for even } n, \end{cases}$$

where $x(0)$ is defined to be some $x_0 > 0$, given at the beginning of Period 1. Hence, in each period, the effort decreases by $v_i(n)\Delta$, where the identity of i depends on who makes the move in that period. Note that $v_i(n)$ is similarly interpreted as the "speed of decline," but its magnitude is twice as large as the counterpart in continuous time, as here each player only contributes half of the time.

As in the continuous-time model, the unknown threshold $c \in [0, x_0)$ has c.d.f. $F(\cdot)$ that satisfies Assumptions 1 and 2. Moreover, the discreteness demands a slightly stronger requirement on the hazard rate:

Assumption 4 *(Strongly Monotone Hazard Rate)*
The inverse hazard rate $\frac{F(\cdot)}{f(\cdot)}$ *is strongly increasing, i.e., for some* $\underline{b} > 0$, $\frac{F(x)}{f(x)} - \frac{F(y)}{f(y)} \geqslant \underline{b}(x - y)$ *for* $0 \leqslant y \leqslant x \leqslant x_0$.

Given the evolution of x, the game terminates at the end of Period $\overline{n}$ if $\overline{n}$ is the smallest $n \geqslant 1$ such that $x(n) \leqslant c$, and $\overline{n}$ can be infinity. In contrast to the continuous-time version, even if the threshold is triggered in Period $\overline{n}$, it is not perfectly inferred; players only know that $c \in [x(\overline{n}), x(\overline{n} - 1))$.

Payoffs come in flows and lumps. Let $r > 0$ be the common real-time discount rate of the players and $\delta \equiv e^{-r\Delta}$ be the discount factor between periods. Define for convenience that $\tilde{\Delta} \equiv \frac{1-e^{-r\Delta}}{r} = \frac{1-\delta}{r}$. In each Period $n \leqslant \overline{n}$, the flow benefit is $\int_0^\Delta e^{-rs} p \mathrm{d}s = p\tilde{\Delta}$ for both players. The flow cost from effort is $\int_0^\Delta e^{-rs} x(n) \mathrm{d}s = x(n)\tilde{\Delta}$ for the player who moves in

Period n and is $x(n-1)\tilde{\Delta}$ for the other player.[9] If the game ever ends with terminal period $\overline{n}$, then the player who moves in that period bears the lumpy cost $L > 0$. Moreover, each player receives a terminal lump sum $\frac{p-x(\overline{n}-1)}{r}$ equalling the present discounted value of a flow $(p - x(\overline{n}-1))\tilde{\Delta}$ per period thereafter.[10]

Recall that in the continuous-time model, the lumpy cost for Player i at the end of the game is $L\frac{v_i(\overline{t})}{V(\overline{t})}$. This can be seen as a limit payoff in the discrete-time game when period length goes to zero. Consider two consecutive periods n and $n+1$, where Player i moves in the former and Player $-i$ moves in the latter. Conditional on the fact that the game ends within the time window $[(n-1)\Delta, (n+1)\Delta]$, it ends during Player i's turn with probability $\frac{v_i(n)\Delta}{(v_i(n)+v_{-i}(n+1))\Delta} = \frac{v_i(n)}{v_i(n)+v_{-i}(n+1)}$. Once we restrict attention to Markov strategies (see below) that are piecewise Lipschitz continuous in x, this ratio converges to the share of loss in the continuous-time model as $\Delta \to 0$.

A pure Markov perfect strategy $\nu_i(\cdot)$ is a mapping from the state space $[0, x_0]$ into the action space $[0, \overline{v}]$, for $i = 1, 2$. Again, we are interested in symmetric MPE in pure strategies. Denote it as $\nu_1(\cdot) = \nu_2(\cdot) = \nu(\cdot)$.

5.1 Existence and Convergence

Having defined the discrete-time model, I present the existence result below, followed by the convergence result that links the discrete- and continuous-time models.

Proposition 6 *Fix a $\Delta > 0$ that is small enough. Suppose that $\overline{v} \geqslant 2\frac{\int_{x_1^*}^{x_0}(F(s)-Lrf(s))\mathrm{d}s}{L\underline{f}}$. There exists a continuum of symmetric pure strategy MPE $E(y_0;\Delta)$, parametrized by $y_0 \in [\underline{y}_0, x_0]$ for some $\underline{y}_0 \in (x_1^*, x_0)$, of the following form:*

$$\nu(x) = \begin{cases} \frac{x-y_{k+1}}{\Delta} & \text{if } x \in (y_{k+1}, y_k] \text{ for some } k \geqslant -1 \\ 0 & \text{if } x \leqslant x_1^* \end{cases},$$

where $y_{-1} \equiv x_0$, $x_1^* < \ldots < y_{k+1} < y_k < \ldots < y_1 < y_0$ and $\lim_{k\to\infty} y_k = x_1^*$.

Proof. See Appendix. ■

[9] This difference in cost results from the asynchronous moves. The difference is unimportant, and it vanishes in the continuous-time model. However, we cannot call $x(n)$ the common effort level. Instead, we call it the frontier of effort.

[10] Here, the imaginary "continuation" payoff is based on the assumption that both players revert to the effort in the last period as the cheapest safe action. Actually, if we allow the game to continue, then this reversion is indeed optimal for a small enough Δ.

Proposition 6 has two facets. First, it claims that there exists some symmetric pure strategy MPE with the *skimming property*, in which there is a strictly decreasing sequence of "critical levels" $\{y_k\}_{k=1}^{\infty}$ of the state variable such that the player currently making a move always chooses an action bringing the state x down to the highest critical level strictly below x. The sequence begins with $y_0 \in [\underline{y}_0, x_0]$ and monotonically converges to x_1^*, and thus, on path, Player 1 reduces the state from x_0 to y_0, Player 2 then brings it to y_1, and again Player 1 sets the new state at y_2, etc. This sequence proceeds all the way down to x_1^* asymptotically. In equilibrium, the current mover facing state y_k is indifferent between remaining at y_k and moving down to y_{k+1}, although the equilibrium requires her to choose the latter.

Second, the proposition also states that there is some indeterminacy leading to multiple equilibria. As is usually the case for equilibrium with the skimming property (in the bargaining problem, for instance), multiplicity arises because unlike other critical levels of the state variable, the initial state may not satisfy the indifference condition. As a result, there is leeway in choosing $y_0 \in [\underline{y}_0, x_0]$ as the first critical point satisfying the indifference condition. Once y_0 is chosen, everything else is determined.

The indeterminacy does not pose a problem in the limit as $\Delta \to 0$, as we will show in Proposition 7. The main idea is that with a smaller Δ, the sequence $\{y_k\}_{k=0}^{\infty}$ is denser, and the interval $[\underline{y}_1, x_0]$ is narrower. Hence, although there is a continuum of equilibria, they differ by less and less as $\Delta \to 0$.

The next proposition shows the convergence of the time paths as $\Delta \to 0$. Thanks to Proposition 6, for any small enough period length $\Delta > 0$, we can select a symmetric pure strategy MPE $E(y_0(\Delta); \Delta)$ indexed by $y_0(\Delta) \in [\underline{y}_0(\Delta), x_0]$, and the resulting path is denoted by $x(\cdot; \Delta)$. Hence, if a sequence $\Delta_n \downarrow 0$, then there exists $N > 0$ s.t. for all $n \geqslant N$, the selection $E(y_0(\Delta_n); \Delta_n)$ exists, generating a sequence of time paths $\{x(\cdot; \Delta_n)\}_{n=N}^{\infty}$.

Proposition 7 *Suppose that* $\overline{v} \geqslant 2\frac{\int_{x_1^*}^{x_0}(F(s)-Lrf(s))\mathrm{d}s}{L\underline{f}}$. *For any fixed sequence* $\Delta_n \downarrow 0$, *there exists* $N > 0$ *s.t. the sequence of paths* $\{x(\cdot; \Delta_n)\}_{n=N}^{\infty}$ *generated by any selection converges to that from the continuous-time model at every* $t \in [0, \overline{t})$.

Proof. See Appendix. ■

Hence, viewed in real time, the path of effort levels in the discrete-time model is a step function, but it converges to a decreasing smooth function, which is the outcome of the continuous-time model. To some extent, this convergence serves as a robustness check of the main model, assuring that both the setup and the equilibrium predictions are valid as a proper limit of some discrete-time model.

Convergence does not always hold for any assumptions on the payoff structure. If, for example, the negative impact of breakdown does not involve a lump sum L but instead costs the entire terminal benefit as an "endogenous punishment," then there are no equilibrium outcomes of the discrete-time version in the vicinity of the continuous-time outcome.

6 Extensions

This section returns to the continuous-time model but considers two extensions. First, I relax the assumption of a constant lumpy cost for all states, allowing for dependence of L on x. Second, I use the ironing approach to solve the problem when Assumption 1 fails.

6.1 Variable Lumpy Cost

It has been previously assumed that the lumpy cost L is fixed. However, one can readily argue that the lumpy cost can vary with the realized threshold. For example, in the maintenance of environment or health, the realized threshold is usually negatively correlated with the cost upon triggering. However, in international relations, a high threshold is often associated with a high lumpy cost, when both are indicators of the "toughness" of the country.

To obtain well-behaved solutions, we need to modify Assumptions 1 through 3:

Assumption 5 *The ratio* $\frac{F(\cdot)}{f(\cdot)L(\cdot)}$ *is strictly increasing.*

Assumption 6 *Both* $f(\cdot)$ *and* $L(\cdot)$ *are uniformly bounded away from zero, i.e.* $\exists \underline{f} > 0$ *s.t.* $\min\{f(c), L(c)\} \geqslant \underline{f}$ *for all* $c \in [0, x_0]$*.*

Assumption 7 *Both* $f(\cdot)$ *and* $L(\cdot)$ *are Lipschitz continuous, i.e.* $\exists \kappa > 0$ *s.t.* $\max\{|f(x) - f(y)|, |L(x) - L(y)|\} \leqslant \kappa |x - y|$ *for all* $x, y \in [0, x_0]$*.*

Accordingly, we need to define $\tilde{x}_1^*$ as the unique solution to $\frac{F(x)}{f(x)L(x)} = r$. The following result is a counterpart of Theorem 1.

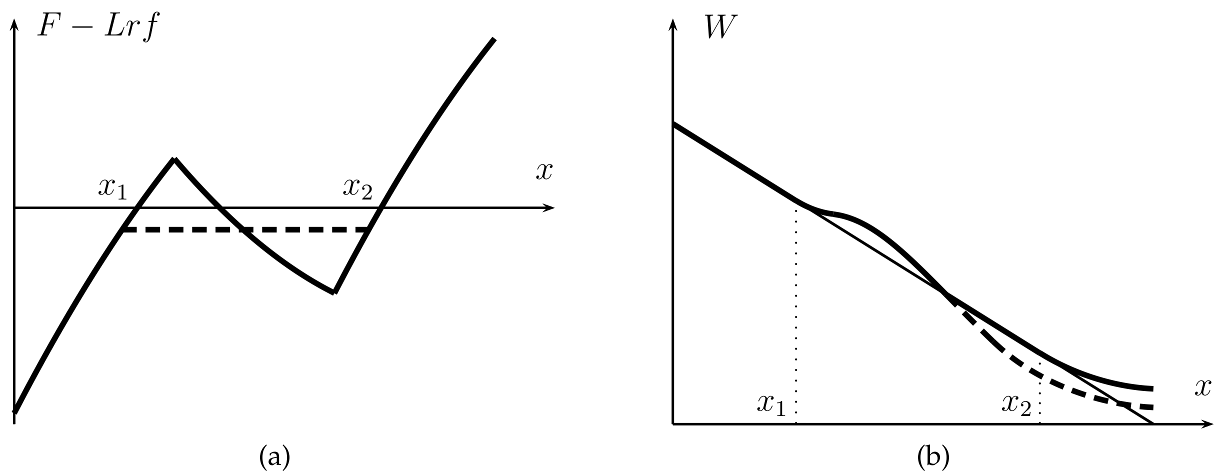

Figure 3: Two candidates. x_1 not worth reaching.

Proposition 8 *Suppose that* $\overline{v} \geqslant \frac{\int_{\tilde{x}_1^*}^{x_0}(F(s)-L(s)rf(s))\mathrm{d}s}{(I-1)\underline{f}^2}$. *There exists a unique symmetric pure strategy stationary Markov equilibrium in the I-player strategic problem. Furthermore, the equilibrium features*

$$\nu(x) = \begin{cases} \frac{\int_{\tilde{x}_1^*}^{x}[F(s)-L(s)rf(s)]\mathrm{d}s}{(I-1)L(x)f(x)} & \text{if } x \in (\tilde{x}_1^*, x_0] \\ 0 & \text{if } x \in (0, \tilde{x}_1^*] \end{cases}$$

$$W(x) = \begin{cases} \frac{p-x}{r} + \frac{1}{rF(x)} \int_{\tilde{x}_1^*}^{x} (F(s) - L(s)rf(s))\, \mathrm{d}s & \text{if } x \in (\tilde{x}_1^*, x_0] \\ \frac{p-x}{r} & \text{if } x \in (0, \tilde{x}_1^*] \end{cases}.$$

The proof of this proposition is similar to that of Theorem 1.

6.2 Non-Monotone Hazard Rate: Ironing

The assumption that $\frac{F(x)}{f(x)}$ is strictly increasing is made for simplicity. However, in some real situations, the distribution function does not necessarily satisfy this condition. For example, the players may know that the subject of experimentation has two major types: tough or delicate, and within each type there are some noises determining the actual realization of the threshold. In this way, the distribution is bimodal, and Assumption 1 may fail. This subsection provides an ironing method that applies when the assumption fails.

In this subsection, I discard both Assumption 1 and the requirement that $\frac{F(x_0)}{f(x_0)} \geqslant Lr$. Due to the failure of the monotone hazard rate, the solutions to $\frac{F(x)}{f(x)} = Lr$ may not be

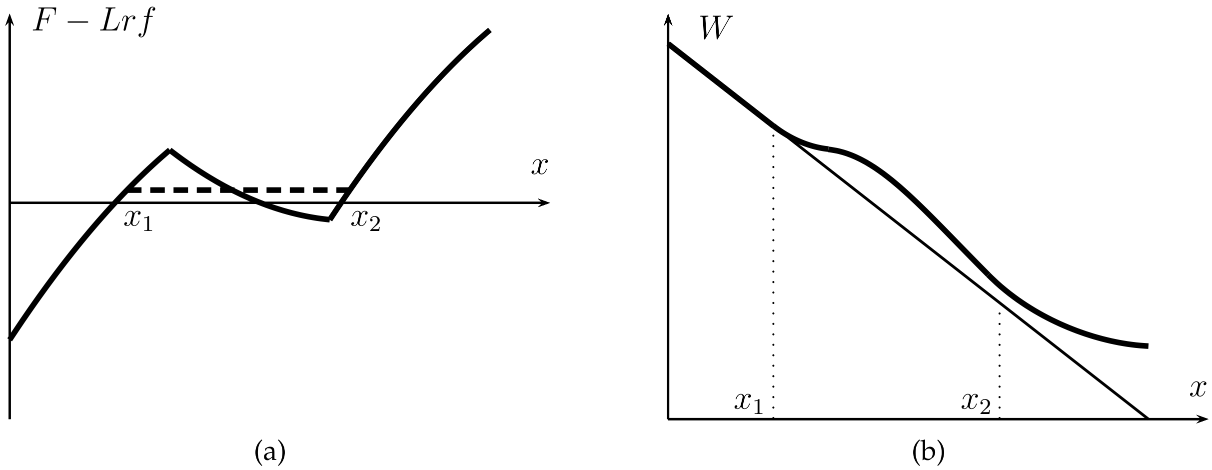

Figure 4: Two candidates. x_1 worth reaching.

uniquely determined or may fail to exist when $\frac{F(x_0)}{f(x_0)} < Lr$.

Figures 3 and 4 are two examples in which there are two candidate cutoff levels x_1 and x_2 at which $F(x) - Lrf(x)$ crosses 0 from below. Intuitively, the distribution F is such that moving down from x_2 is myopically unprofitable due to the high density f near x_2, but further reducing effort becomes profitable when the players overcome the initial hardship. The tradeoff the group of players face is whether it is worthwhile to move from x_2 all the way down to x_1. The left panel of Figure 3 depicts a situation in which the total area under the curve from x_1 to x_2 is negative. After ironing, the curve to the left of x_2 becomes negative everywhere, meaning that x_1 is not the correct candidate for the cutoff effort level. The right panel shows the hypothetical value functions if the players use x_1 (dashed curve) or x_2 (solid curve) as the cutoff effort. If x_1 were the cutoff, then the value at x_2 would lie below the lower bound $\frac{p-x_2}{r}$, which can be guaranteed by remaining at x_2 forever, implying the suboptimality of stopping at x_1. In this case, the forward-looking players choose to stop at x_2 because the cost outweighs the benefit. On the contrary, Figure 4 depicts a situation in which the ironed curve crosses 0 at x_1, meaning that the initial hardship is worth suffering for a subsequent reward.

Formally, define $K(x) \equiv \int_0^x [F(s) - Lrf(s)]\mathrm{d}s$ and $\overline{K}(\cdot) \equiv \text{Vex}(K(\cdot))$ as the convexification of $K(\cdot)$. With this transformation, $\overline{K}'(x)$ is weakly increasing. The result below shows the selection criterion when facing a non-monotone hazard rate.

Proposition 9 *(i) If $\overline{K}'(x_0) \leqslant 0$, then any player in the symmetric equilibrium should stay at x_0 forever.*

(ii) If $\overline{K}'(x_0) > 0$ and $\overline{K}'(x)$ has a unique intersection with 0 at $\overline{x}^*$, then it is the unique cutoff

effort when $\overline{v}$ is sufficiently large.

Proof. See Appendix. ■

7 Conclusion

In a dynamic game with multiple players experimenting on an unknown threshold, I characterized the unique symmetric stationary Markov equilibrium in pure strategies. In this equilibrium, players slowly contribute to decreasing the effort level, which rests asymptotically at some cutoff level above zero if breakdown does not occur. The equilibrium dynamics depend on the extent of information externalities, the prior distribution of the threshold, the severity of breakdown, the number of players, and patience. The hazard rate of reaching the threshold declines over time. These main results offer a possible explanation to the various economic phenomena mentioned in the introduction, and generate some testable implications.

A discrete-time version of the game was studied, with a presentation of the existence and convergence results. As a robustness check, this served as an interpretation of the continuous-time game as the limit of a sequence of discrete-time games, when period length tends to zero.

I considered two extensions. The first extension allowed for a state-dependent lumpy cost. The second extension used the ironing method to overcome the non-monotone hazard rate problem.

One interesting, yet difficult, ingredient that is not covered by the paper concerns a changing threshold. If one allows the threshold to evolve according to some stochastic process, then even if players trigger a breakdown, the continuation play is non-trivial, and there could be possibility of multiple breakdowns. This better fits some economic stories in which "history repeats itself," and is left for future research.

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A Appendix

A.1 Proof of Proposition 2

Proof. We prove part (a) first. If an equilibrium requires $\mathcal{V}(x) = 0$ for some $x \in (x_1^*, x_0]$, then HJB implies that $W_i(x) = \frac{p-x}{r}$. Also, $W_i(x') \geqslant \frac{p-x'}{r}$ for all $x' < x$, so that $W_i'(x) \leqslant -\frac{1}{r}$. Plug these into the first order condition to see that

$$\frac{f(x)}{F(x)}\left(\frac{p-x}{r} - W_i(x) - L\right) - W_i'(x) \geqslant \frac{F(x) - Lrf(x)}{rF(x)} > 0$$

violating optimality.

Now we turn to part (b). Suppose that there is a positive measured set $A \subset (0, x_I^*)$ such that $\mathcal{V}(x) > 0$ for all $x \in A$. For $x \in A$ and for every $i = 1, \ldots, I$, solve $W_i'(x)$ from the HJB (7) and we have

$$W_i'(x) = -L\frac{\nu_i(x)}{\mathcal{V}(x)}\frac{f(x)}{F(x)} - \left(W_i(x) - \frac{p-x}{r}\right)\left(\frac{f(x)}{F(x)} + \frac{r}{\mathcal{V}(x)}\right)$$

Since Player i always has the choice to take null action $v_i = 0$, we must require $W_i(x) \geqslant \frac{p-x}{r}$, then the above becomes

$$W_i'(x) \leqslant -L\frac{\nu_i(x)}{\mathcal{V}(x)}\frac{f(x)}{F(x)} < -\frac{1}{r}\frac{I\nu_i(x)}{\mathcal{V}(x)}$$

Adding up the above for all i, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{i=1}^{I} W_i(x) < -\frac{I}{r}$$

On the other hand, for $x \in (0, x_I^*)\backslash A$, we have $\mathcal{V}(x) = 0$ and $\sum_{i=1}^{I} W_i(x) = -\frac{I(p-x)}{r}$. So,

$$
\begin{aligned}
& \lim_{x \to 0} \sum_{i=1}^{I} W_i(x) \\
= \; & \sum_{i=1}^{I} W_i(x_I^*) - \lim_{x \to 0} \int_x^{x_I^*} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=1}^{I} W_i(s) \mathrm{d}s \\
= \; & \sum_{i=1}^{I} W_i(x_I^*) - \lim_{x \to 0} \left(\int_A \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=1}^{I} W_i(s) \mathrm{d}s + \int_{(0,x_I^*)\backslash A} \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=1}^{I} W_i(s) \mathrm{d}s \right) \\
> \; & \frac{I(p - x_I^*)}{r} + \lim_{x \to 0} \frac{I(x_I^* - x)}{r} \\
= \; & \frac{Ip}{r}
\end{aligned}
$$

However, this consists a contradiction since $\lim_{x \to 0} \sum_{i=1}^{I} W_i(x) = \frac{Ip}{r}$ by Sandwich Theorem. Hence the measure of A is zero and $W_i(x) = \frac{p-x}{r}$. Plugging back $W_i(x)$ to the FOC implies that A is empty. ∎

A.2 Proof of Proposition 3

Proof. Part (a) is directly implied by Proposition 2. We now turn to part (b). Suppose there is positive-measured set $A \subset (0, x_1^*)$ such that $\nu(x) > 0$ for all $x \in A$. Then for $x \in A$, first order condition for Player i reads

$$
\frac{f(x)}{F(x)} \left(\frac{p - x}{r} - W_i(x) - L \right) - W_i'(x) \geqslant 0
$$

so that

$$
W_i'(x) \leqslant \frac{f(x)}{F(x)} \left(\frac{p - x}{r} - W_i(x) - L \right) < -\frac{1}{r}
$$

On the other hand, for $x \in (0, x_1^*)\backslash A$, we have $\nu(x) = 0$ and $W_i(x) = -\frac{p-x}{r}$. Also,

$W_i(x') \geqslant \frac{p-x'}{r}$ for all $x' < x$, so that $W_i'(x) \leqslant -1/r$. Hence,

$$\begin{aligned}\lim_{x\to 0} W_i(x) &= W_i(x_1^*) - \lim_{x\to 0}\int_x^{x_1^*} W_i'(s)\mathrm{d}s \\ &= W_i(x_1^*) - \lim_{x\to 0}\left(\int_A W_i'(s)\mathrm{d}s + \int_{(0,x_1^*)\setminus A} W_i'(s)\mathrm{d}s\right) \\ &> \frac{p-x_1^*}{r} + \lim_{x\to 0}\frac{x_1^*-x}{r} \\ &= \frac{p}{r}\end{aligned}$$

which violates the boundary condition at $x = 0$.

This contradiction implies that the measure of A is zero, and furthermore that A is empty. ■

A.3 Proof of Theorem 1

Proof. That the proposed expression (9) for $\nu(\cdot)$ consists a symmetric equilibrium is guaranteed by verification Theorem taking other players' strategies as fixed.

The uniqueness of symmetric pure strategy equilibrium is shown below by following Tarski's fixed point Theorem. Notice first that combining (7) and (8) we have

$$BR_i(\nu_{-i})\begin{cases} = \overline{v} & \text{if } W_i(x) > \frac{p-x}{r} + \nu_{-i}(x)\frac{f(x)}{F(x)}\frac{L}{r}, \\ \in [0,\overline{v}] & \text{if } W_i(x) = \frac{p-x}{r} + \nu_{-i}(x)\frac{f(x)}{F(x)}\frac{L}{r}, \\ = 0 & \text{if } W_i(x) < \frac{p-x}{r} + \nu_{-i}(x)\frac{f(x)}{F(x)}\frac{L}{r}\end{cases}. \tag{11}$$

For any Lipschitz continuous function W with $-\frac{1}{r} \leqslant W' \leqslant 0$, define functional $\psi_1(\cdot)$ by $(\psi_1(W))(x) \equiv \min\left\{(I-1)\overline{v}, \frac{(rW(x)-(p-x))F(x)}{Lf(x)}\right\}$, and define functional $\psi_2(\nu_{-i})$ as the value function of a player when the aggregate action of other players is ν_{-i}. From (11), we know that W is a value function of a player in a symmetric pure strategy equilibrium if and only if W is a fixed point of $\psi \equiv \psi_2 \circ \psi_1$.

Evidently, ψ_1 is non-decreasing in W. The following lemma shows that ψ_2 is non-decreasing.

Lemma 3 *Consider problem* (7). *If $\tilde{\nu}_{-i} \geqslant \nu_{-i}$ for all x, then the respective solutions satisfy $\tilde{W}_i(x) \geqslant W_i(x)$ for all x.*

Proof. First, we bound $W_i(x)$ from below by $U_1(x)$. If not, then $\exists x_1$ s.t. $W_i(x_1) < U_1(x)$ and $W_i'(x_1) < U_1'(x_1)$. Following similar logic of Keller, Rady and Cripps (2005), we have the following inequality:

$$\begin{aligned}
& \max_{v_i \in [0,\overline{v}]} (p - x_1) + \frac{v_i f(x_1)}{F(x_1)} \left(\frac{p - x_1}{r} - U_1(x_1) - L \right) - VU_1'(x_1) \\
= \; & rU_1(x_1) \\
> \; & rW_i(x_1) \\
= \; & \max_{v_i \in [0,\overline{v}]} (p - x_1) + \frac{v_i f(x_1)}{F(x_1)} \left(\frac{p - x_1}{r} - W_i(x_1) - L \right) - v_i W_i'(x_1) \\
& + \frac{\nu_{-i} f(x_1)}{F(x_1)} \left(\frac{p - x_1}{r} - W_i(x_1) \right) - \nu_{-i} W_i'(x_1) \\
\geqslant \; & \max_{v_i \in [0,\overline{v}]} (p - x_1) + \frac{v_i f(x_1)}{F(x_1)} \left(\frac{p - x_1}{r} - U_1(x_1) - L \right) - v_i U_1'(x_1) \\
& + \frac{\nu_{-i} f(x_1)}{F(x_1)} \left(\frac{p - x_1}{r} - U_1(x_1) \right) - \nu_{-i} U_1'(x_1) \\
\geqslant \; & \max_{v_i \in [0,\overline{v}]} (p - x_1) + \frac{v_i f(x_1)}{F(x_1)} \left(\frac{p - x_1}{r} - U_1(x_1) - L \right) - v_i U_1'(x_1)
\end{aligned}$$

a contradiction. The last inequality follows from the fact that $\frac{f(x_1)}{F(x_1)} \left(\frac{p-x_1}{r} - U_1(x_1) \right) - U_1'(x_1) \geqslant 0$.

Next, we need to show that $\frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) \right) - W_i'(x) \geqslant 0$.

If $v_i > 0$, then the FOC of (7) must require that $\frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) - L \right) - W_i'(x) \geqslant 0$, hence we have $\frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) \right) - W_i'(x) > 0$. If $v_i = 0$ and $\nu_{-i} = 0$, then $W_i(x) = \frac{p-x}{r}$ and $W_i'(x) \leqslant -\frac{1}{r}$, hence $\frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) \right) - W_i'(x) \geqslant \frac{1}{r} > 0$. If $v_i = 0$ and $\nu_{-i} > 0$, then $\frac{f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) \right) - W_i'(x) = \frac{r}{\nu_{-i}} \left(W_i(x) - \frac{p-x}{r} \right) \geqslant 0$. In sum, the required inequality is true.

Finally,

$$\begin{aligned}
& \tilde{W}_i(x) \\
= \; & \max_{v_i \in [0,\overline{v}]} (p - x) + \frac{v_i f(x)}{F(x)} \left(\frac{p - x}{r} - \tilde{W}_i(x) - L \right) - v_i \tilde{W}_i'(x) \\
& + \frac{\tilde{\nu}_{-i} f(x)}{F(x)} \left(\frac{p - x}{r} - \tilde{W}_i(x) \right) - \tilde{\nu}_{-i} \tilde{W}_i'(x) \\
\geqslant \; & \max_{v_i \in [0,\overline{v}]} (p - x) + \frac{v_i f(x)}{F(x)} \left(\frac{p - x}{r} - \tilde{W}_i(x) - L \right) - v_i \tilde{W}_i'(x) \\
& + \frac{\nu_{-i} f(x)}{F(x)} \left(\frac{p - x}{r} - \tilde{W}_i(x) \right) - \nu_{-i} \tilde{W}_i'(x)
\end{aligned}$$

Since $\tilde{W}_i(0) = W_i(0) = (p-x)/r$, if ever $\tilde{W}_i < W_i$, there must exist x_1 s.t. $\tilde{W}_i(x_1) < W_i(x_1)$ and $\tilde{W}_i'(x_1) < W_i'(x_1)$. Then

$$\begin{aligned}
& \tilde{W}_i(x_1) \\
\geqslant \ & \max_{v_i \in [0,\overline{v}]} (p-x) + \frac{v_i f(x)}{F(x)} \left(\frac{p-x}{r} - \tilde{W}_i(x) - L \right) - v_i \tilde{W}_i'(x) \\
& + \frac{\nu_{-i} f(x)}{F(x)} \left(\frac{p-x}{r} - \tilde{W}_i(x) \right) - \nu_{-i} \tilde{W}_i'(x) \\
\geqslant \ & \max_{v_i \in [0,\overline{v}]} (p-x) + \frac{v_i f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) - L \right) - v_i W_i'(x) \\
& + \frac{\nu_{-i} f(x)}{F(x)} \left(\frac{p-x}{r} - W_i(x) \right) - \nu_{-i} W_i'(x) \\
= \ & W_i(x_1)
\end{aligned}$$

a contradiction. ■

Hence, it follows from Tarski's fixed point Theorem that $\psi = \psi_2 \circ \psi_1$ has minimal and maximal fixed points W_- and W_+. For W_-, we know from Proposition 3 that

$$W_-'(x) = \begin{cases} -\frac{1}{r} & \text{if } x \in [0, x_1^*] \\ \min\{\frac{p-x-rW_-(x)}{I\overline{v}} + \frac{(I-1)Lf(x)}{IF(x)}, 0\} - \frac{(-p+Lr+x+rW_-(x))f(x)}{rF(x)} & \text{if } x \in [x_1^*, x_0] \end{cases}$$

Similarly, there exists a unique cutoff $x_+ \in [0, x_1^*]$ for W_+. Hence the difference $\overline{W} \equiv W_+ - W_-$ satisfies

$$\overline{W}'(x) = \begin{cases} 0 & \text{if } x \in [0, x_1^*] \\ -\frac{\overline{W}(x) f(x)}{F(x)} + \min\{\frac{p-x-rW_+(x)}{I\overline{v}} + \frac{(I-1)Lf(x)}{IF(x)}, 0\} & \\ \quad - \min\{\frac{p-x-rW_-(x)}{I\overline{v}} + \frac{(I-1)Lf(x)}{IF(x)}, 0\} & \text{if } x \in (x_1^*, x_0] \end{cases}.$$

Therefore, $\overline{W}(x) \geqslant 0$ and $\overline{W}'(x) \leqslant 0$ for all $x \in [0, x_0]$. Meanwhile, $\overline{W}(0) = 0$ because $W_+(0) = W_-(0) = \frac{p}{r}$. So $\overline{W}(x) = 0$ for all $x \in [0, x_0]$, implying uniqueness. ■

A.4 Proof of Proposition 5

Proof. Since $\lim_{x \downarrow x_1^*} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{F(x)}{f(x)} \right) = b$, we have $\lim_{x \downarrow x_1^*} \nu(x) / \frac{b(x-x_1^*)^2}{2L(I-1)} = 1$. For any $A > 1$, there exists a $\hat{x}$ s.t. $\frac{1}{A} \frac{b(x-x_1^*)^2}{2L(I-1)} < \nu(x) < A \frac{b(x-x_1^*)^2}{2L(I-1)}$ for $x \in [x_1^*, \hat{x}]$. Denote $\hat{t}$ as the solution to $\hat{x} = x(\hat{t})$. Starting from the point $(\hat{t}, \hat{x})$, the path $x(t)$ is sandwiched by paths where $\nu(x)$ is replaced by $\frac{1}{A}\frac{b(x-x_1^*)^2}{2L(I-1)}$ and $A\frac{b(x-x_1^*)^2}{2L(I-1)}$, respectively. Hence, we have

$$\frac{2(I-1)Lt(\hat{x}-x_1^*)}{2(I-1)L+AbI(t-\hat{t})(\hat{x}-x_1^*)}<\frac{x(t)-x_1^*}{1/t}<\frac{2(I-1)Lt(\hat{x}-x_1^*)}{2(I-1)L+A^{-1}bI(t-\hat{t})(\hat{x}-x_1^*)}$$

for $t>\hat{t}$, so that

$$\limsup_{t\to\infty}\frac{x(t)-x_1^*}{1/t}\leqslant\frac{2(I-1)LA}{bI}$$
$$\liminf_{t\to\infty}\frac{x(t)-x_1^*}{1/t}\geqslant\frac{2(I-1)L}{AbI}$$

Since the above hold for any $A>1$, we know $\lim_{t\to\infty}\frac{x(t)-x_1^*}{1/t}=\frac{2(I-1)L}{bI}$. ■

A.5 Proof of Lemma 1

Proof. Notice that x_1^* is a function of L and r. Taking derivative of $\nu(x)$ w.r.t. L and r gives

$$\frac{\partial\nu(x)}{\partial L}=-\frac{\int_{x_1^*}^{x}F(s)\mathrm{d}s}{(I-1)L^2f(x)}-\frac{\partial x_1^*}{\partial L}\frac{F(x_1^*)-Lrf(x_1^*)}{(I-1)Lf(x)}=-\frac{\int_{x_1^*}^{x}F(s)\mathrm{d}s}{(I-1)L^2f(x)}<0$$
$$\frac{\partial\nu(x)}{\partial r}=-\frac{F(x)-F(x_1^*)}{(I-1)f(x)}-\frac{\partial x_1^*}{\partial r}\frac{F(x_1^*)-Lrf(x_1^*)}{(I-1)Lf(x)}=-\frac{F(x)-F(x_1^*)}{(I-1)f(x)}<0$$

Since both $\frac{1}{I-1}$ and $\frac{I}{I-1}$ are decreasing in I, the statement for I is obvious. ■

A.6 Proof of Proposition 6

Proof. The key step of the proof is Lemma 4 below. In order to understand the structure of MPE's in Proposition 6, consider the following system of difference equations with generic variables y_k and z_k.

$$\begin{aligned}(p-y_k)\tilde{\Delta}+\delta z_k=&(p-y_{k+1})\tilde{\Delta}-L\frac{F(y_k)-F(y_{k+1})}{F(y_k)}\\&+\delta\left[z_{k+1}\frac{F(y_{k+1})}{F(y_k)}+\frac{p-y_k}{r}\frac{F(y_k)-F(y_{k+1})}{F(y_k)}\right],\end{aligned}\tag{12}$$

$$\begin{aligned}z_k=&(p-y_k)\tilde{\Delta}\\&+\delta\left[[(p-y_{k+1})\tilde{\Delta}+\delta z_{k+1}]\frac{F(y_{k+1})}{F(y_k)}+\frac{p-y_k}{r}\frac{F(y_k)-F(y_{k+1})}{F(y_k)}\right],\end{aligned}\tag{13}$$

$y_0\in[x_1^*,x_0]$ is given. (14)

One can think of $\{y_k\}_{k=0}^{\infty}$ as a sequence of critical effort levels and $\{z_k\}_{k=0}^{\infty}$ as the sequence of corresponding payoffs when faced with the critical effort levels and the player is currently *not* the mover. The MPE we look for has the skimming property such that starting from a state $x \in (y_{k+1}, y_k]$, the current mover takes action $\frac{x-y_{k+1}}{\Delta}$ to bring down the state to the highest critical level strictly below x, namely y_{k+1}. Equation (12) is the indifference condition that facing state $x = y_k$, the player is indifferent between staying at y_k and moving one step down to y_{k+1}, given the continuation play prescribed by the MPE. Equation (13) is the promise keeping condition saying that the payoff of the non-mover facing state y_k is the weighted sum of current payoff and continuation payoff, where the indifference condition is already embodied in the continuation payoff. Now we state Lemma 4.

Lemma 4 *Suppose Δ is small. There exists a unique $z_0 \geqslant 0$ such that the solution to the difference equation system* (12)-(14) *has the property that y_k monotonically decreases and $\lim_{k\to\infty} y_k = x_1^*$.*

Proof.

Step 1: Change of variables: $z_k = \frac{u_k}{F(y_k)} + \frac{p-y_k}{r}$.

Noting also that $\tilde{\Delta} = (1-\delta)/r$, the system is simplified to

$$u_{k+1} = \frac{ru_k - \delta(y_k - y_{k+1})F(y_{k+1})}{\delta^2 r} \tag{15}$$

$$\delta[Lr - (1-\delta)(y_k - y_{k+1})][F(y_k) - F(y_{k+1})] = (1-\delta^2)ru_k \tag{16}$$

Now, (15), (16) and (14) consist a new difference equation system with generic variables y_k and u_k. For a fixed y_0, there is one-to-one mapping from u_0 to z_0, so we want to find the unique u_0 such that y_k monotonically converges to x_1^*. We can immediately rule out the case $u_0 < 0$. To see why, note that $y_0 - y_1 < x_0 - x_1^*$ is bounded, so for δ close enough to 1 (Δ small), $Lr - (1-\delta)(y_0 - y_1) > 0$, and hence from (16) we have $u_0 \geqslant 0$.

Step 2: Induction on the new system.

First we examine conditions for (y_k, u_k) s.t. (y_{k+1}, u_{k+1}) exists as the unique solution to (15) and (16). From (16) define

$$J(y_k, y_{k+1}, u_k) \equiv \delta[Lr - (1-\delta)(y_k - y_{k+1})][F(y_k) - F(y_{k+1})] - (1-\delta^2)ru_k$$

so that

$$\frac{\partial J}{\partial y_{k+1}} = (1-\delta)[F(y_k) - F(y_{k+1})] - [Lr - (1-\delta)(y_k - y_{k+1})]f(y_{k+1})$$
$$\frac{\partial J}{\partial y_k} = -(1-\delta)[F(y_k) - F(y_{k+1})] + [Lr - (1-\delta)(y_k - y_{k+1})]f(y_k)$$
$$\frac{\partial J}{\partial u_k} = -(1-\delta^2)r$$

Now, $J(y_k, y_k, u_k) \leqslant 0$ and moreover, $\frac{\partial J}{\partial y_{k+1}} < 0$ for $y_{k+1} \in [x_1^*, y_k]$ when $y_k > x_1^*$ and δ is close enough to 1 (remember that f is bounded below by $\underline{f} > 0$). By Intermediate Value Theorem, there exists a unique $y_{k+1} \in [x_1^*, y_k]$ if and only if $J(y_k, x_1^*, u_k) \geqslant 0$, i.e. u_k is not too large given y_k. Having pinned down y_{k+1}, we immediately determine u_{k+1} by (15).

The following lines show that $u_k \geqslant 0 \Rightarrow u_{k+1} \geqslant 0$ if $y_{k+1} \in [x_1^*, y_k]$ exists.

$$\begin{aligned}
\delta^2 r u_{k+1} &= r u_k - \delta(y_k - y_{k+1})F(y_{k+1}) \geqslant r u_k - \frac{\delta}{\underline{f}}[F(y_k) - F(y_{k+1})]F(y_{k+1}) \\
&= r u_k \left(1 - \frac{(1-\delta^2)F(y_{k+1})}{\underline{f}[Lr - (1-\delta)(y_k - y_{k+1})]}\right) \\
&\geqslant r u_k \left(1 - \frac{1-\delta^2}{\underline{f}[Lr - (1-\delta)x_0]}\right) \geqslant 0
\end{aligned}$$

where the last inequality follows when δ is close to 1.

With the above observations, we define the transition function from tuple to tuple:

$$\Gamma(y_k, u_k) \equiv (\Gamma_y(y_k, u_k), \Gamma_u(y_k, u_k)) \equiv (y_{k+1}, u_{k+1})$$

where (y_{k+1}, u_{k+1}) is the solution to (15) (16) satisfying $y_{k+1} \in [x_1^*, y_k]$, if it exists. Otherwise, the function returns some arbitrary vector, say $(-1, -1)$, to indicate nonexistence. $\Gamma^{(n)} \equiv (\Gamma_y^{(n)}(y_k, u_k), \Gamma_u^{(n)}(y_k, u_k))$ is the function Γ applied n times.

Step 3: Uniform upper bound on u_0 such that $\Gamma_y^{(\infty)}(y_0, u_0) \geqslant x_1^*$.

Fix (y_0, u_0), we want to bound the locus of $\Gamma^n(y_0, u_0)$ from below on the $y - u$ plane. Define a lower bound function:

$$G_L(y) \equiv u_0 - \frac{1}{\delta r}\int_y^{y_0} [F(s) - Lrf(s)]\mathrm{d}s \tag{17}$$

where $y_0 \geqslant x_1^*$, with $G_L(y_0) = u_0$. If we can show that $u_k \geqslant G_L(y_k) \Rightarrow \Gamma_u(y_k, u_k) \geqslant$

$G_L(\Gamma_y(y_k, u_k))$ provided $\Gamma_y(y_k, u_k) \in [x_1^*, y_k]$, then we have $\Gamma_u^{(n)}(y_k, u_k) \geqslant G_L(\Gamma_y^{(n)}(y_k, u_k))$ for all n by induction, provided $\Gamma_y^{(n)}(y_k, u_k) \in [x_1^*, \Gamma_y^{(n-1)}(y_k, u_k)]$. Let $u_k = G_L(y_k)+\varepsilon$ where $\varepsilon \geqslant 0$. Denote $y_{k+1} = \Gamma_y(y_k, u_k)$ and $u_{k+1} = \Gamma_u(y_k, u_k) = \frac{ru_k - \delta(y_k - y_{k+1})F(y_{k+1})}{\delta^2 r}$ (by (15)). The claim above is true if

$$\frac{r[G_L(y_k)+\varepsilon] - \delta(y_k - y_{k+1})F(y_{k+1})}{\delta^2 r} \geqslant G_L(y_{k+1}) = G_L(y_k) - \frac{1}{\delta r}\int_{y_{k+1}}^{y_k} [F(s) - Lrf(s)]\mathrm{d}s$$

$$\Leftrightarrow \quad (1-\delta^2)G_L(y_k) \geqslant -\frac{\delta}{r}\int_{y_{k+1}}^{y_k} [F(s) - F(y_{k+1}) - Lrf(s)]\mathrm{d}s - \varepsilon \tag{18}$$

On the other hand, (16) gives

$$G_L(y_k) = u_k - \varepsilon = \frac{\delta}{(1-\delta^2)r}[Lr - (1-\delta)(y_k - y_{k+1})][F(y_k) - F(y_{k+1})] - \varepsilon$$

so this together with (18) yields

$$(1-\delta)(y_k - y_{k+1})[F(y_k) - F(y_{k+1})] - \int_{y_{k+1}}^{y_k} [F(s) - F(y_{k+1})]\mathrm{d}s \leqslant 0 \tag{19}$$

because it holds for all $\varepsilon \geqslant 0$.

Let $f_m \equiv \min_{x\in[y_{k+1}, y_k]} f(x) \geqslant \underline{f}$, then $F(s) - F(y_{k+1}) \geqslant f_m(s - y_{k+1})$. By Lipschitz continuity of f, we have $\frac{F(y_k)-F(y_{k+1})}{y_k - y_{k+1}} \leqslant f_m + \kappa(y_k - y_{k+1}) \leqslant f_m + \kappa x_0$. So, a sufficient condition for (19) is

$$(1-\delta)(y_k - y_{k+1})^2(f_m + \kappa x_0) - f_m\int_{y_{k+1}}^{y} (s - y_{k+1})\mathrm{d}s \leqslant 0$$

$$\Leftrightarrow \quad \delta \geqslant 1 - \frac{f_m}{2(f_m + \kappa x_0)}$$

Since $f_m \geqslant \underline{f}$, a uniform sufficient condition is $\delta \geqslant 1 - \frac{\underline{f}}{2(\underline{f}+\kappa x_0)}$.

Note that $G_L(\cdot)$ is strictly increasing in $[x_1^*, y_0]$. Also, $G_L(x_1^*) = u_0 - \frac{1}{\delta r}\int_{x_1^*}^{y_0} F(s)\mathrm{d}s + \frac{L}{\delta}[F(y_0) - F(x_1^*)] > 0$ if $u_0 > \overline{u} \equiv \frac{y_0 - x_1^*}{\delta r}$.

Hence, if u_0 is too big for a given y_0, then $u_k > G_L(y_k) \geqslant G_L(x_1^*) > 0$ whenever $y_k \in [x_1^*, x_0]$. However, in order to have $y_\infty = x_1^*$, we must have $u_\infty = 0$ by (16), a contradiction. Therefore, given any $y_0 \in [x_1^*, x_0]$, u_0 has a uniform upper bound independent of δ. Moreover, because $G_L'(y) > 0$ for all $y \in [x_1^*, y_0]$, we can restrict attention to points $(y, u) \in [x_1^*, y_0] \times [0, \overline{u}]$ for further analysis.

Step 4: Uniform lower bound on u_0 such that $\Gamma_y^{(\infty)}(y_0, u_0) \leqslant x_1^*$.

Fix (y_0, u_0), we want to bound the locus of $\Gamma^n(y_0, u_0)$ from above on the $y - u$ plane. To achieve this, define

$$G_H(y) = u_0 + a(1-\delta)(y_0 - y) - \frac{1}{\delta r}\int_y^{y_0}[F(s) - Lrf(s)]\mathrm{d}s \tag{20}$$

where $y_0 \geqslant x_1^*$, a is a positive constant to be determined later, and $G_H(y_0) = u_0$. Because of induction, we only need to show that $u \leqslant G_H(y) \Rightarrow \Gamma_u(y, u) \leqslant G_H(\Gamma_y(y, u))$ if $\Gamma_y(y, u) \in [x_1^*, y]$ and $u \in [0, \overline{u}]$. I claim that this is true for some $a > 0$. Let $u_k = G_H(y_k) - \varepsilon$ where $\varepsilon \geqslant 0$. Denote $y_{k+1} = \Gamma_y(y_k, u_k)$ and $u_{k+1} = \Gamma_u(y_k, u_k) = \frac{ru_k - \delta(y_k - y_{k+1})F(y_{k+1})}{\delta^2 r}$ as before. The claim is true if

$$\begin{aligned} & \frac{r[G_H(y_k) - \varepsilon] - \delta(y_k - y_{k+1})F(y_{k+1})}{\delta^2 r} \\ \leqslant \quad & G_H(y_{k+1}) = G_H(y_k) + a(1-\delta)(y_k - y_{k+1}) - \frac{1}{\delta r}\int_{y_{k+1}}^{y_k}[F(s) - Lrf(s)]\mathrm{d}s \end{aligned}$$

which is true if the following sufficient condition is satisfied:

$$(1-\delta^2)G_H(y_k) \leqslant a\delta^2(1-\delta)(y_k - y_{k+1}) - \frac{\delta}{r}\int_{y_{k+1}}^{y_k}[F(y_k) - F(y_{k+1}) - Lrf(s)]\mathrm{d}s + \varepsilon \tag{21}$$

On the other hand, (16) gives

$$G_H(y_k) = u_k + \varepsilon = \frac{\delta}{(1-\delta^2)r}[Lr - (1-\delta)(y_k - y_{k+1})][F(y_k) - F(y_{k+1})] + \varepsilon$$

so that (21) reduces to

$$\begin{aligned} & a \geqslant \frac{[F(y_k) - F(y_{k+1})](y_k - y_{k+1}) - r\varepsilon}{r(1-\delta)(y_k - y_{k+1})} \text{ for all } \varepsilon \geqslant 0 \\ \Rightarrow \quad & a \geqslant \frac{(1+\delta)u_k}{\delta[Lr - (1-\delta)(y_k - y_{k+1})]} \end{aligned}$$

For δ close to 1, it is sufficient to set $a = \frac{4u_0}{Lr}$. With this value of a, $u_0 = \max_{y \in [x_1^*, y_0]}$, and G_H is indeed an upper bound for the sequence $\Gamma^{(n)}(y_0, u_0)$ when δ is close to 1.

Note that

$$\begin{aligned} G_H(x_1^*) &= u_0 + a(1-\delta)(y_0 - x_1^*) - \frac{1}{\delta r}\int_{x_1^*}^{y_0} f(s)\left(\frac{F(s)}{f(s)} - \frac{F(x_1^*)}{f(x_1^*)}\right)\mathrm{d}s \\ &\leqslant u_0 + a(1-\delta)(y_0 - x_1^*) - \frac{1}{\delta r}\int_{x_1^*}^{y_0} \underline{f}(s - x_1^*)\underline{b}\mathrm{d}s \\ &= u_0 + \frac{4u_0}{Lr}(1-\delta)(y_0 - x_1^*) - \frac{(y_0 - x_1^*)^2\underline{f}\underline{b}}{2\delta r} \end{aligned}$$

so that when δ is close to 1, $G_H(x_1^*) < 0$ if $u_0 < \frac{(y_0 - x_1^*)^2\underline{f}\underline{b}}{4r}$. In order to have $y_\infty = x_1^*$, we require $u_\infty = 0$, but the point $(x_1^*, 0)$ is above the graph of G_H, a contradiction. Actually, if $u_0 < \frac{(y_0 - x_1^*)^2\underline{f}\underline{b}}{4r}$, we know that for δ close to 1, $J(y_0, x_1^*, u_0) > 0$, so that $y_1 \in (x_1^*, y_0]$. By induction, $y_k \in (x_1^*, y_0]$ for any k, meaning that $\{y_k\}_{k=0}^{\infty}$ is a decreasing sequence with lower bound x_1^*, admitting a limit $y_\infty \in (x_1^*, y_0]$. Therefore, fix any $y_0 \in [x_1^*, x_0]$, a too small u_0 leads to $y_\infty > x_1^*$.

Step 5: Order preserving property of $\Gamma(\cdot,\cdot)$.

Order preserving property means that for any two different points $(y,u),(y',u') \in [x_1^*, x_0] \times [0, \overline{u}]$ with $y' \leqslant y$ and $u' \geqslant u$, $\Gamma_y(y',u') < \Gamma_y(y,u)$ and $\Gamma_u(y',u') > \Gamma_u(y,u)$, provided $\Gamma_y(y',u'), \Gamma_y(y,u) \geqslant x_1^*$.

We want to prove the above for $u, u' <$. Given existence, recall from Step 2 that $\frac{\partial J}{\partial y_{k+1}} < 0$, $\frac{\partial J}{\partial y_k} > 0$ and $\frac{\partial J}{\partial u_k} < 0$ for δ close to 1. By Implicit Function Theorem, $\frac{\partial y_{k+1}}{\partial y_k} > 0$ and $\frac{\partial y_{k+1}}{\partial u_k} < 0$.

Moreover, from (15) we use the chain rule to get

$$\frac{\partial u_{k+1}}{\partial u_k} = \frac{1}{\delta^2} - \frac{1}{\delta r}[(y_k - y_{k+1})f(y_{k+1}) - F(y_{k+1})]\frac{\partial y_{k+1}}{\partial u_k} \tag{22}$$

Note that while $(y_k - y_{k+1})f(y_{k+1}) - F(y_{k+1})$ is bounded, $\frac{\partial y_{k+1}}{\partial u_k}$ uniformly converges to zero as $\delta \to 1$, so $\frac{\partial y_{k+1}}{\partial u_k} > 0$ for δ close to 1.

Finally,

$$\frac{\partial u_{k+1}}{\partial y_k} = -\frac{F(y_{k+1})}{\delta r} - \frac{1}{\delta r}[(y_k - y_{k+1})f(y_{k+1}) - F(y_{k+1})]\frac{\partial y_{k+1}}{\partial y_k}$$

If $\frac{\partial y_{k+1}}{\partial y_k} \leqslant 1$, then $\frac{\partial u_{k+1}}{\partial y_k} < 0$. Otherwise, $\frac{\partial y_{k+1}}{\partial y_k} < \frac{Lrf(y_k)-(1-\delta)}{Lrf(y_{k+1})-(1-\delta)}$ when δ is close to 1. By the assumption on the hazard rate, we have

$$\begin{aligned} & \frac{F(y_k)}{f(y_k)} > \frac{F(y_{k+1})}{f(y_{k+1})} + \underline{b}(y_k - y_{k+1}) \\ \Rightarrow \;\; & F(y_{k+1})[f(y_k) - f(y_{k+1})] < f(y_{k+1})[F(y_k) - F(y_{k+1})] - \underline{b}(y_k - y_{k+1})f(y_k)f(y_{k+1}) \end{aligned}$$

so that

$$\begin{aligned} \frac{\partial u_{k+1}}{\partial y_k} &< \frac{f(y_{k+1})(y_k - y_{k+1})}{\delta r[f(y_{k+1})Lr - (1-\delta)]} \left[\left(\frac{F(y_k) - F(y_{k+1})}{y_k - y_{k+1}} - f(y_k) - \underline{b} f(y_k) \right) Lr + (1-\delta) \right] \\ &< \frac{f(y_{k+1})(y_k - y_{k+1})}{\delta r[f(y_{k+1})Lr - (1-\delta)]} \left[\left(\kappa(y_k - y_{k+1}) - \underline{b} f(y_k) \right) Lr + (1-\delta) \right] \\ &< \frac{f(y_{k+1})(y_k - y_{k+1})}{\delta r[f(y_{k+1})Lr - (1-\delta)]} \left[\left(\frac{\kappa r \overline{u}(1-\delta^2)}{\delta[Lr - (1-\delta)(y_k - y_{k+1})]} - \underline{b} f(y_k) \right) Lr + (1-\delta) \right] \\ &< -\frac{\underline{b} f(y_k - y_{k+1})}{2r} \end{aligned}$$

for δ close to 1.

With the signs of the four partial derivatives, we have

$$\Gamma_y(y', u') < \Gamma_y(y, u), \;\; \Gamma_u(y', u') > \Gamma_u(y, u)$$

if $u' \geqslant u$, $y' \leqslant y$ and $(y', u') \neq (y, u)$. Iteration forward gives us the desired ordering for the whole sequence.

Step 6: For those u_0 s.t. $y_\infty \in [x_1^*, y_0]$, y_∞ is strictly decreasing in u_0.

Suppose this is not true, then there exist two initial points (y_0, u_0) and (y_0, u_0') with $u_0' > u_0$ but $y_\infty' = y_\infty \geqslant x_1^*$ (by Step 5, $y_\infty' > y_\infty$ is impossible). We will show that this cannot happen.

The idea is that if $y_\infty' = y_\infty$ then $y_k - y_k'$ will be small for any k, contradicting the initial difference $F(y_1) - F(y_1') > 0$. Formally, for any $s \geqslant 0$, by (16)

$$\begin{aligned}
& F(y_s') - F(y_{s+1}') \\
= \quad & \frac{Lr - (1-\delta)(y_s - y_{s+1})}{Lr - (1-\delta)(y_s' - y_{s+1}')}[F(y_s) - F(y_{s+1})]\frac{u_s'}{u_s} \\
> \quad & \frac{Lr - (1-\delta)(y_s - y_{s+1})}{Lr - (1-\delta)(y_s' - y_{s+1}')}[F(y_s) - F(y_{s+1})] \\
> \quad & [F(y_s) - F(y_{s+1})]\left(1 - \frac{(1-\delta)(y_s - y_s')}{Lr - (1-\delta)x_0}\right)
\end{aligned}$$

using the fact that $u_s' > u_s$ and $y_s > y_s'$.

For an arbitrary $k \geqslant 1$, adding up the above inequality on the far left and far right sides for s from k to ∞, we have

$$\begin{aligned}
& F(y_k') - F(y^*) \\
> \quad & F(y_k) - F(y^*) - \frac{1-\delta}{Lr - (1-\delta)x_0}\sum_{s=k}^{\infty}[F(y_s) - F(y_{s+1})](y_s - y_s') \\
> \quad & F(y_k) - F(y^*) - \frac{1-\delta}{Lr - (1-\delta)x_0}x_0 \qquad (23)
\end{aligned}$$

which means

$$\begin{aligned}
& F(y_k) - F(y_k') < \frac{1-\delta}{Lr - (1-\delta)x_0}x_0 \\
\Rightarrow \quad & y_k - y_k' < Ax_0 \qquad (24)
\end{aligned}$$

where $A \equiv \frac{1-\delta}{\underline{f}[Lr-(1-\delta)x_0]}$.

Note that in deriving (23) we use the initial bound that $y_s - y_s' < x_0$ for all s, and arrive at another bound that that $y_k - y_k' < Ax_0$ for all k. One can iterate between inequalities (23) and (24) for arbitrarily many (n) rounds to get $y_k - y_k' < A^n x_0$ for all k. The factor A is smaller than 1 if δ is close to 1, resulting in $y_k - y_k' = 0$. But we know from Step 5 that for $u_0' > u_0$, it must be the case that $y_k > y_k'$. This contradiction means that $y_\infty' < y_\infty$.

Step 7: There is a unique u_0 such that $\Gamma_y^{(\infty)}(y_0, u_0) = x_1^*$.

For any given $y_0 \in [x_1^*, x_0]$, define $\mathcal{U} \equiv \{u_0 \in [0, \overline{u}] : \Gamma_y^{(\infty)}(y_0, u_0) \in [x_1^*, y_0]\}$. From Steps 3, 4 and 5 we know that this set is non-empty, bounded above, and has the property $u \in \mathcal{U} \Rightarrow u' \in \mathcal{U}$ for all $u' \in [0, u]$. Hence, $\mathcal{U}$ is an interval $[0, u_0^*]$ or $[0, u_0^*)$ for some $u_0^* > 0$.

It can be shown that $u_0^* = \sup \mathcal{U} \in \mathcal{U}$. To see this, let $u_0 > u_0^*$ so that $\Gamma_y^{(\infty)}(y_0, u_0) < x_1^*$, and hence there exists a smallest $k \geqslant 1$ s.t. $\Gamma_y^{(k)}(y_0, u_0) < x_1^*$. This means $J(y_{k-1}, x_1^*, u_{k-1}) < 0$. Notice that fix y_0, $(y_{k-1}, u_{k-1}) = \Gamma^{(k-1)}(y_0, u_0)$ is a continuous function of u_0, so if $u_0' < u_0$ is close enough to u_0, we still have $J(y_{k-1}', x_1^*, u_{k-1}') < 0$. Therefore, $\mathcal{U}$ is closed and $u_0^* \in \mathcal{U}$.

It remains to show that $\Gamma_y^{(\infty)}(y_0, u_0^*) = x_1^*$. Suppose $\Gamma_y^{(\infty)}(y_0, u_0^*) = \hat{x} > x_1^*$, towards contradiction. Let $(y_k, u_k) = \Gamma^{(k)}(y_0, u_0^*)$. Fix an arbitrary $\eta > 0$. Because of the convergence, there is a K such that $|y_K - \hat{x}| < \frac{\eta}{2}$ and $0 < u_K < \frac{\eta}{2}$. On the other hand, because of the continuity of $\Gamma^{(K)}(\cdot, \cdot)$, we know that there is a $\varepsilon > 0$ small enough s.t. $|y_K' - y_K| < \frac{\eta}{2}$ and $|u_K' - u_K| < \frac{\eta}{2}$, where $(y_K', u_K') = \Gamma^{(K)}(y_0, u_0^* + \varepsilon)$. Hence, $|y_K' - \hat{x}| < \eta$ and $0 < u_K' < \eta$. By virtue of Step 4, if $u_K' < \frac{(y_K' - x_1^*)^2 \underline{f} b}{4r}$ then $\Gamma^{(\infty)}(y_{k_1}', u_{k_1}')$ exists. This condition is satisfied when the arbitrarily fixed η is small enough. Step 4 then implies $u_0^* + \varepsilon \in \mathcal{U}$, contradicting the definition of u_0^*. Therefore, $\Gamma_y^{(\infty)}(y_0, u_0^*) = x_1^*$. By Step 6, it is the unique u_0 s.t. $y_\infty = x_1^*$.
■

From above we know that every $y_0 \in [x_1^*, x_0]$ pins down a unique $u_0^*(y_0)$. In the following, we aim to show that for any $y_0 \in [\underline{y}_0, x_0]$, the strategy defined in Proposition 6 with $y_k = \Gamma_y^{(k)}(y_0, u_0^*(y_0))$ consists an MPE, where $\underline{y}_0 \equiv \Gamma_y(x_0, u_0^*(x_0))$. For any $x \in (x_1^*, x_0]$, we must have $x \in (y_k, y_{k-1}]$ for some $k \geqslant 0$.

Step 1: Verify that for every $k \geqslant 0$, the payoff of the current mover facing state $x \in (y_k, y_{k-1}]$ and brings new state x' is decreasing in x' on every interval (y_{k+s}, y_{k+s-1}) for all $s \geqslant 1$ and on interval (y_k, x), where $x' = x - v\Delta$ and $y_{-1} = x_0$.

The payoff of the current mover if she brings new state $x' \in (y_{k+s}, y_{k+s-1})$ is

$$\begin{aligned} U(x'; x) &= (p - x')\tilde{\Delta} - L\frac{F(x) - F(x')}{F(x)} \\ &\quad + \delta\left[\frac{F(x) - F(x')}{F(x)}\frac{p - x}{r} + \frac{F(x')}{F(x)}w(x')\right] \end{aligned}$$

where

$$\begin{aligned} w(x') &= (p - x')\tilde{\Delta} \\ &\quad + \delta\left[\frac{F(x') - F(y_{k+s})}{F(x')}\frac{p - x'}{r} + \frac{F(y_{k+s})}{F(x')}[(p - y_{k+s})\tilde{\Delta} + \delta z_{k+s}]\right] \end{aligned}$$

and $z_{k+s} = \frac{u_{k+s}}{F(y_{k+s})} + \frac{p - y_{k+s}}{r}$, $(y_{k+s}, u_{k+s}) = \Gamma^{(k+s)}(y_0, u_0^*)$.

Plugging in $\tilde{\Delta} = (1-\delta)/r$, we have

$$\begin{aligned} & \frac{\mathrm{d}U(x';x)}{\mathrm{d}x} > 0 \\ \Leftrightarrow \quad & -\delta F(x') + \delta^2 F(y_{k+s}) + (Lr + \delta(x - x'))]f(x) > 0 \end{aligned}$$

which is true when δ is close enough to 1. So, for $x \in (y_{k+s}, y_{k+s-1}]$, $U(x';x)$ is increasing in x'. The above analysis also holds for $x' \in (y_k, x)$. In particular, $U(x';x) \leqslant U(x;x)$ when $x' \in (y_k, x]$.

Step 2: Verify that the current mover facing state $x \in (y_k, y_{k-1}]$ prefers y_k to x.

The equilibrium action leading to y_k yields

$$\begin{aligned} U(y_k;x) &= (p - y_k)\tilde{\Delta} - L\frac{F(x) - F(y_k)}{F(x)} \\ &\quad + \delta\left[\frac{F(x) - F(y_k)}{F(x)}\frac{p - M}{r} + \frac{F(y_k)}{F(x)}z_k\right] \end{aligned}$$

so

$$\begin{aligned} & U(y_k;x) \geqslant U(x;x) \\ \Leftrightarrow \quad & -(1-\delta^2)\delta r u_k + [Lr - (1-\delta)(x - y_k)]F(x) - [Lr + (1-\delta)\delta(x - y_k)]F(y_k) \leqslant 0 \end{aligned} \tag{25}$$

where $u_k = (z_k - (p - y_k)/r)[F(y_k) - F(m)]$. Recall from (15) and (16) that (25) holds with equality when $x = y_{k-1}$ (provided $k \geqslant 1$). Taking the derivative of the left hand side of (25) w.r.t. x gives

$$-(1-\delta)[F(x) + \delta F(y_k)] + [Lr - (1-\delta)(x - y_k)]f(x) > 0$$

when δ is close to 1. That means, the left hand side of (25) is non-positive for $x \in (y_k, y_{k-1}]$, i.e. $U(y_k;x) \geqslant U(x;x)$ when $k \geqslant 1$.

The only caveat is for the case $k = 0$. Because x_0 does not belong to the sequence $\{y_k\}_{k=0}^{\infty}$, the indifference condition does not hold at $x = x_0$. In order to have $U(y_0;x) \geqslant U(x;x)$ when $x \in (y_0, x_0]$, we need the condition $y_0 \geqslant \Gamma_y(x_0, u_0^*(x_0)) = \underline{y}_0$. In sum, the incentive conditions are satisfied if $y_0 \geqslant \underline{y}_0$.

Step 3: Verify that the current mover prefers moving to y_k to all actions below.

Step 1 has shown that for any $s \geqslant 0$, setting new state within (y_{k+s+1}, y_{k+s}) is dominated by setting y_{k+s}, so we only need to show that the mover prefers y_{k+s} to y_{k+s+1} for any $s \geqslant 0$. To see this, note

$$
\begin{aligned}
& U(y_{k+s}; x) \geqslant U(y_{k+s+1}; x) \\
\Leftrightarrow \quad & \delta r(u_{k+s} - u_{k+s+1}) - (1-\delta)(y_{k+s} - y_{k+s+1})F(x) \\
& +[Lr + \delta(x - y_{k+s})]F(y_{k+s}) - [Lr + \delta(x - y_{k+s+1})]F(y_{k+s+1}) \geqslant 0
\end{aligned}
$$

which holds with equality when $x = y_{k+s}$ by (15) and (16). Taking the derivative of the left hand side w.r.t. x gives

$$
\begin{aligned}
& \delta[F(y_{k+s}) - F(y_{k+s+1})] - (1-\delta)(y_{k+s} - y_{k+s+1})f(x) \\
\geqslant \quad & [\delta \underline{f} - (1-\delta)f(x)](y_{k+s} - y_{k+s+1}) \\
> \quad & 0
\end{aligned}
$$

when δ is close to 1 (because of the Lipschitz continuity of f). Hence for any $s \geqslant 0$, $U(y_{k+s}; x) \geqslant U(y_{k+s+1}; x)$ whenever $x \geqslant y_{k+s}$, and by telescoping $U(y_k; x) \geqslant U(y_{k+s}; x)$ for any $s \geqslant 1$.

The three steps above confirm that choosing the new state at y_k is indeed globally optimal when $x \in [y_k, y_{k-1}]$, $\forall k \geqslant 0$, and hence the proposed strategy profile consists a MPE. Moreover, given any $y_0 \in [\underline{y}_0, x_0]$, the sequence $\Gamma^{(n)}(y_0, u_0^*(y_0))$ uniquely pins down a MPE. Hence there is a continuum of MPEs. ■

A.7 Proof of Proposition 7

Proof. We prove the theorem by construction. First we prove that

$$
\lim_{\Delta \to 0} \left(\frac{y_{\lfloor \frac{t}{\Delta} \rfloor + 1} - y_{\lfloor \frac{t}{\Delta} \rfloor}}{\Delta} + \frac{\int_{x^*}^{y_{\lfloor \frac{t}{\Delta} \rfloor}} \big([F(s) - F(m)] - 2Lrf(s) \big) \mathrm{d}s}{Lf(y_{\lfloor \frac{t}{\Delta} \rfloor})} \right) = 0
$$

To see this, note from (16) that

$$
\begin{aligned}
y_{k+1}-y_k &\geqslant -\frac{(1-\delta^2)ru_k}{\delta[Lr-(1-\delta)x_0][f(y_k)-\kappa(y_k-y_{k+1})]} \\
&\geqslant -\frac{(1-\delta^2)ru_k}{\delta[Lr-(1-\delta)x_0][f(y_k)-(1-\delta)B]}, \qquad (26)\\
y_{k+1}-y_k &\leqslant -\frac{(1-\delta^2)ru_k}{\delta Lr[f(y_k)+\kappa(y_k-y_{k+1})]} \\
&\leqslant -\frac{(1-\delta^2)ru_k}{\delta Lr[f(y_k)+(1-\delta)B]} \qquad (27)
\end{aligned}
$$

where $B=\frac{2r\bar{u}\kappa}{Lr\underline{f}}$.

Meanwhile, note from (15) that $u_{k+1}-u_k=\frac{1-\delta^2}{\delta^2}u_k+\frac{1}{\delta r}(y_{k+1}-y_k)F(y_{k+1})$ so

$$
\begin{aligned}
&\frac{F(y_{k+1})-Lr[f(y_k)+(1-\delta)B]}{\delta r} \\
\leqslant\ & \frac{u_{k+1}-u_k}{y_{k+1}-y_k}=\frac{1-\delta^2}{\delta^2(y_{k+1}-y_k)}u_k+\frac{1}{\delta r}F(y_{k+1}) \\
\leqslant\ & \frac{F(y_{k+1})-[Lr-(1-\delta)x_0][f(y_k)-(1-\delta)B]}{\delta r}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{s=0}^{\infty}\frac{F(y_{k+s+1})-Lr[f(y_{k+i})+(1-\delta)B]}{\delta r}(y_{k+s}-y_{k+s+1}) \\
\leqslant\ & u_k=u_\infty-\sum_{s=0}^{\infty}(u_{k+s+1}-u_{k+s}) \\
\leqslant\ & \sum_{s=0}^{\infty}\frac{F(y_{k+s+1})-[Lr-(1-\delta)x_0][f(y_{k+s})-(1-\delta)B]}{\delta r}(y_{k+s}-y_{k+s+1}).
\end{aligned}
$$

Plugging the above into (26) and (27), we have

$$\frac{y_{\lfloor \frac{t}{\Delta}\rfloor} - y_{\lfloor \frac{t}{\Delta}\rfloor+1}}{\tilde{\Delta}} \leqslant \frac{r(1+\delta)}{\delta^2[Lr-(1-\delta)x_0][f(y_{\lfloor \frac{t}{\Delta}\rfloor})-(1-\delta)B]} \times$$
$$\sum_{s=0}^{\infty}\Big[F(y_{\lfloor \frac{t}{\Delta}\rfloor+s+1}) - [Lr-(1-\delta)x_0][f(y_{\lfloor \frac{t}{\Delta}\rfloor+s})-(1-\delta)B]\Big](y_{\lfloor \frac{t}{\Delta}\rfloor+s} - y_{\lfloor \frac{t}{\Delta}\rfloor+s+1}), \quad (28)$$
$$\frac{y_{\lfloor \frac{t}{\Delta}\rfloor} - y_{\lfloor \frac{t}{\Delta}\rfloor+1}}{\tilde{\Delta}} \leqslant \frac{r(1+\delta)}{\delta^2 Lr[f(y_{\lfloor \frac{t}{\Delta}\rfloor})+(1-\delta)B]} \times$$
$$\sum_{s=0}^{\infty}\Big[F(y_{\lfloor \frac{t}{\Delta}\rfloor+s+1}) - Lr[f(y_{\lfloor \frac{t}{\Delta}\rfloor+s})+(1-\delta)B]\Big](y_{\lfloor \frac{t}{\Delta}\rfloor+s} - y_{\lfloor \frac{t}{\Delta}\rfloor+s+1}). \quad (29)$$

which can be further simplified to

$$\frac{2}{Lf(y_{\lfloor \frac{t}{\Delta}\rfloor})}\sum_{s=0}^{\infty}[F(y_{\lfloor \frac{t}{\Delta}\rfloor+s}) - Lrf(y_{\lfloor \frac{t}{\Delta}\rfloor+s})](y_{\lfloor \frac{t}{\Delta}\rfloor+s} - y_{\lfloor \frac{t}{\Delta}\rfloor+s+1}) + C_H(1-\delta) - g(y_{\lfloor \frac{t}{\Delta}\rfloor})$$
$$\geqslant \frac{y_{\lfloor \frac{t}{\Delta}\rfloor} - y_{\lfloor \frac{t}{\Delta}\rfloor+1}}{\tilde{\Delta}} - g(y_{\lfloor \frac{t}{\Delta}\rfloor})$$
$$\geqslant \frac{2}{Lf(y_{\lfloor \frac{t}{\Delta}\rfloor})}\sum_{s=0}^{\infty}[F(y_{\lfloor \frac{t}{\Delta}\rfloor+s}) - Lrf(y_{\lfloor \frac{t}{\Delta}\rfloor+s})](y_{\lfloor \frac{t}{\Delta}\rfloor+s} - y_{\lfloor \frac{t}{\Delta}\rfloor+s+1}) - C_L(1-\delta) - g(y_{\lfloor \frac{t}{\Delta}\rfloor})$$

for some constants C_L and C_H, when δ is close to 1. Also, $g(y) \equiv 2\int_{x_1^*}^{y} \frac{F(s)-Lrf(s)}{Lf(y)}\mathrm{d}s$. Note that since $F(x) - Lrf(x)$ is integrable and $y_k - y_{k+1}$ uniformly converges to 0, the left and right hand side both converge to 0 by Dominated Convergence Theorem, and the convergence is uniform in t. By Sandwich Theorem, we know that

$$\lim_{\Delta\to 0}\left(\frac{y_{\lfloor \frac{t}{\Delta}\rfloor} - y_{\lfloor \frac{t}{\Delta}\rfloor+1}}{\tilde{\Delta}} - g(y_{\lfloor \frac{t}{\Delta}\rfloor})\right) = 0,$$

uniformly for all $t \geqslant 0$. For a fixed t, $\lfloor \frac{t}{\Delta}\rfloor$ is finite, so $g(y_{\lfloor \frac{t}{\Delta}\rfloor}) > 0$. Also, $\lim_{\Delta\to 0}(\tilde{\Delta} - \Delta) = \lim_{\Delta\to 0}\frac{1-e^{-r\Delta}}{r} - \Delta$=0. The convergence therefore can be rewritten as $\lim_{\Delta\to 0}\frac{y_{\lfloor \frac{t}{\Delta}\rfloor} - y_{\lfloor \frac{t}{\Delta}\rfloor+1}}{g\left(y_{\lfloor \frac{t}{\Delta}\rfloor}\right)\Delta} = 1$. The convergence is now uniform for all times $t' \leqslant t$, meaning that by Sandwich Theorem again,

$$\lim_{\Delta\to 0}\left(\left\lfloor\frac{t}{\Delta}\right\rfloor\right)^{-1}\sum_{k=0}^{\lfloor \frac{t}{\Delta}\rfloor-1}\frac{y_k - y_{k+1}}{g(y_k)\Delta} = 1$$
$$\Rightarrow \lim_{\Delta\to 0}\sum_{k=0}^{\lfloor \frac{t}{\Delta}\rfloor-1}\frac{y_k - y_{k+1}}{g(y_k)} = t$$

On the other hand, by Dominated Convergence Theorem we have

$$0 = \lim_{\Delta \to 0} \left(\sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \frac{y_k - y_{k+1}}{g(y_k)} - \int_{y_{\lfloor \frac{t}{\Delta} \rfloor}}^{x_0} \frac{1}{g(s)} \mathrm{d}s \right) = t - \lim_{\Delta \to 0} \int_{y_{\lfloor \frac{t}{\Delta} \rfloor}}^{x_0} \frac{1}{g(s)} \mathrm{d}s.$$

Note that the state at real time t is $x(t;\Delta) = y_{\lfloor \frac{t}{\Delta} \rfloor}$, so that

$$\lim_{\Delta \to 0} \int_{x(t;\Delta)}^{x_0} \frac{1}{g(s)} \mathrm{d}s = \int_{\lim_{\Delta \to 0} x(t;\Delta)}^{x_0} \frac{1}{g(s)} \mathrm{d}s = t.$$

Comparing the above with the time path in continuous time: $\int_{x(t)}^{x_0} \frac{1}{2\nu(s)} \mathrm{d}s = t$, and noting that $g(s) = 2\nu(s)$, we immediately have $\lim_{\Delta \to 0} x(t;\Delta) = x(t)$. ■

A.8 Proof of Proposition 9

Proof.

Part (a) Suppose in a symmetric Markov equilibrium players bring down the effort level below x_0, then there exists a $\hat{x} \in [0, x_0)$ such that $\hat{x} = \inf\{x : \nu(y) > 0 \text{ for all } y \in [x, x_0]\}$. When $x \in (\hat{x}, x_0)$, the first order condition for a player reads

$$\frac{f(x)}{F(x)} \left(\frac{p - x}{r} - W(x) - L \right) - W'(x) \geqslant 0,$$

so that

$$W'(x) \leqslant \frac{f(x)}{F(x)} \left(\frac{p - x}{r} - W(x) - L \right) < -\frac{1}{r},$$

which translates into

$$W(x) \leqslant W(\hat{x}) - \frac{x_0 - \hat{x}}{r} + \int_{\hat{x}}^{x_0} (F(s) - Lrf(s)) \mathrm{d}s.$$

Note that $W(\hat{x}) = \frac{p - \hat{x}}{r}$ and

$$\int_{\hat{x}}^{x_0} (F(s) - Lrf(s)) \mathrm{d}s = \int_{\hat{x}}^{x_0} K'(s) \mathrm{d}s = K(x_0) - K(\hat{x}) \leqslant \overline{K}(x_0) - \overline{K}(\hat{x}) = \int_{\hat{x}}^{x_0} \overline{K}'(s) \mathrm{d}s < 0.$$

Hence, $W(x_0) < \frac{p - x_0}{r}$, lower than the no-move payoff, a contradiction.

Part (b) If $\overline{x}^*$ is the unique root of $\overline{K}'$, then $\overline{K}$ has a strict global minimum at $\overline{x}^*$, which must also be a strict global minimum of K. In other words, $\overline{x}^*$ is a root of the original function K', and $K(\overline{x}^*) = \overline{K}(\overline{x}^*)$.

If we replace x_0 in Part (a) with $\overline{x}^*$, then we immediately know that players should not move down below $\overline{x}^*$.

On the other hand, we want to show that players will move down if $x > \overline{x}^*$. If not, then $W(x) = \frac{p-x}{r}$. Consider a deviation for a player: moving at full speed until reaching $\overline{x}^*$. Note that the worst case in such a deviation is that no other player moves, but even that yields a higher payoff than staying at x forever. To see why, the worst deviation payoff is

$$\hat{W}(x) = \frac{p-x}{r} + \frac{1}{rF(x)} \int_{\overline{x}^*}^{x} \left(F(s) - Lrf(s)\right) e^{-r(x-s)/\overline{v}} \mathrm{d}s$$

where

$$\begin{aligned}
& \int_{\overline{x}^*}^{x} \left(F(s) - Lrf(s)\right) e^{-r(x-s)/\overline{v}} \mathrm{d}s \\
= \;& \int_{\overline{x}^*}^{x} K'(s) e^{-r(x-s)/\overline{v}} \mathrm{d}s \\
= \;& K(x) - e^{-r(x-\overline{x}^*)/\overline{v}} K(\overline{x}^*) - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x} K(s) e^{-r(x-s)/\overline{v}} \mathrm{d}s \\
\geqslant \;& \overline{K}(x) - e^{-r(x-\overline{x}^*)/\overline{v}} \overline{K}(\overline{x}^*) - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x} \overline{K}(s) e^{-r(x-s)/\overline{v}} \mathrm{d}s - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x} [K(s) - \overline{K}(s)] e^{-r(x-s)/\overline{v}} \mathrm{d}s \\
= \;& \int_{\overline{x}^*}^{x} \overline{K}'(s) e^{-r(x-s)/\overline{v}} \mathrm{d}s - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x} [K(s) - \overline{K}(s)] e^{-r(x-s)/\overline{v}} \mathrm{d}s \\
\geqslant \;& \frac{1}{2} \int_{\overline{x}^*}^{x} \overline{K}'(s) \mathrm{d}s - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x} [K(s) - \overline{K}(s)] \mathrm{d}s
\end{aligned}$$

for $\overline{v}$ greater than some $\overline{v}_1 > 0$. Since $\overline{x}^*$ is the unique root of $\overline{K}'$, there exists an interval $(\overline{x}^*, \tilde{x})$ s.t. $K(x) = \overline{K}(x)$ for all $x \in (\overline{x}^*, \tilde{x})$. Hence, for $x \leqslant \tilde{x}$ we have

$$\frac{1}{2} \int_{\overline{x}^*}^{x} \overline{K}'(s) \mathrm{d}s - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x} [K(s) - \overline{K}(s)] \mathrm{d}s > 0.$$

For $\tilde{x} < x \leqslant x_0$, we have

$$\begin{aligned}
& \frac{1}{2} \int_{\overline{x}^*}^{x} \overline{K}'(s) \mathrm{d}s - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x} [K(s) - \overline{K}(s)] \mathrm{d}s \\
> \;& \frac{1}{2} \int_{\overline{x}^*}^{\tilde{x}} \overline{K}'(s) \mathrm{d}s - \frac{r}{\overline{v}} \int_{\overline{x}^*}^{x_0} [K(s) - \overline{K}(s)] \mathrm{d}s \\
> \;& 0
\end{aligned}$$

when

$$\overline{v} > \overline{v}_2 \equiv \frac{2r\frac{r}{\overline{v}}\int_{\overline{x}^*}^{x_0}[K(s) - \overline{K}(s)]\mathrm{d}s}{\int_{\overline{x}^*}^{\tilde{x}}\overline{K}'(s)\mathrm{d}s}.$$

In sum, when $\overline{v} > \max\{\overline{v}_1, \overline{v}_2\}$, there is a profitable deviation for a player, i.e., stopping above $\overline{x}^*$ is not an equilibrium. ■